AN ITERATIVE EIGENDECOMPOSITION APPROACH TO BLIND SOURCE SEPARATION

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ABSTRACT

In this work we address the generalized eigendecomposition approach (GED) to the blind source separation problem. We present an alternative formulation for GED based on the definition of congruent pencils. Making use of this definition, and matrix block operations, the eigendecompostion approach to blind source separation is completely characterized. We also present an iterative method to compute the eigendecomposition of a symmetric positive definite pencil.

1. INTRODUCTION

The Blind Source Signal Separation is a problem that arises in many application areas such as communications, speech and biomedical signal processing. The objective is to extract the source signals from some sensor measurements. Generally, it is assumed that each measured (or mixed) signal is an instantaneous mixture of the source signals. The extraction must be carried on without knowing the structure of the linear combination (the mixing matrix) and the source signals. The mathematical model for this problem is

$$\mathbf{y}(t) = A\mathbf{s}(t)$$

Where A is the mixing matrix, $\mathbf{y}(t)$ and $\mathbf{s}(t)$ are vectors of mixed and source signals at time t, respectively.

Most of the solutions, for blind source separation, comprise two steps[1][3]. In the first step, called the whitening (sphering) phase, the measured data is linearly transformed such that the correlation matrix of the output vector equals the identity matrix. This linear transformation is usually computed using the standard eigendecomposition of the mixed data correlation matrix. During this phase the dimensionality of the measured vector is also reduced to the dimension of the source vector. After that, the separation matrix, between the whitening data and the output, is an orthogonal matrix which is computed applying different strategies. In algorithms like AMUSE and EFOBI [1] a standard eigendecomposition is performed in a matrix derived from fourth-order cumulant or time-delayed correlation definitions. The global separation matrix, or an estimate of the inverse of A, is the product of the two matrices computed on the two phases of the method.

Recently, the problem was also addressed as a generalized eigendecomposition (GED). The solution comprises the simultaneous diagonalization of a matrix pencil $(R_{x1}$, $Rx_2)$ computed in the mixed signals. These matrices are calculated with different strategies: Souloumiac [5] consider two segments of signals with distinct energy; Lo [7] considers different embedding spaces of the chaotic signals; and Molgedey [8], Chang [4] computes time-delayed correlation matrices and Tomé [6] considers filtered versions of the mixed signals. Using this method the separation matrix, i.e. the matrix that simultaneous diagonalizes the pencil, is the transpose of eigenvector matrix of the generalized eigendecomposition of pencil.

In this work, we will formulate the GED method to blind source separation using a linear algebra approach based on the definition of congruent pencils [10]. The use of congruent pencil definition and of block matrix operations constitute a very simple formulization of the GED approach to the blind source separation. We also review methods that perform the eigendecomposition of a matrix pencil based on two consecutive standard eigendecompositions. We will introduce an iterative algorithm to compute the GED of a symmetric matrix pencil. The algorithm is based on the power method and deflation techniques, to perform standard eigendecompositions, and on the use of the spectral factorization of a matrix to approximate a linear transformation. This method can be an on-line algorithm for blind source separation if the matrix pencil can be computed iteratively.

2. THE GENERALIZED EIGENDECOMPOSITION APPROACH

The generalized eigendecomposition formulatio of the blind source separation problem is based on the relation of two pencils: the source matrix pencil (R_{s1}, R_{s2}) and the mixed pencil $(R_{x1},\ R_{x2})$. The two pencils are called congruent

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pencils [10] if there exists an invertible matrix A, such that

$$R_{x1} = AR_{s1}A^T \text{ and } \quad R_{x2} = AR_{s2}A^T \tag{1}$$

In the blind source separation problem the matrix A is the instantaneous mixing matrix and can be an $m \times n$ matrix, i.e., the number of mixed signals (m) is not equal the number of source signals (n). In that case we will show that the properties which characterize congruent pencils are also applied if $m > n$. Therefore, the inverse (or the pseudo-inverse) of the mixing matrix can be estimated, using the mixed pencil, if the eigenvector matrix of the source pencil is diagonal. The following propositions constitute the required prove.

Proposition 1 *: Congruent pencils have the same eigenvalues.*

The eigenvalues of a pencil are the roots of the characteristic polynomial, $\chi(\lambda)$,

$$\chi(\lambda) = \det(R_{x1} - \lambda R_{x2}) = 0 \tag{2}$$

If A is an invertible matrix, then

$$\det(R_{x1} - \lambda R_{x2}) = \det(A)\det(R_{s1} - \lambda R_{s2})\det(A^T),$$

which as the same roots as characteristic polynomial of the source matrix pencil

$$\chi(\lambda) = \det(R_{s1} - \lambda R_{s2}) = 0 \tag{3}$$

When A is a rectangular matrix $(m > n)$, if A^TA is an invertible matrix, the congruent pencil $(A^TAR_{s1}A^TA$, $A^TAR_{s2}A^TA)$ has also the same eigenvalues, and so the mixed pencil (with $m \times m$ matrices) has also n eigenvalues equal to the eigenvalues of the source pencil.

Proposition 2 *: The eigenvectors of the source pencil are related with the eigenvectors of the mixed pencil*

The generalized eigendecomposition statement of the mixed pencil

$$R_{x1}E = R_{x2}ED \tag{4}$$

where E is the eigenvector matrix and it will be an unique matrix (with the columns normalized to unity length) if the diagonal matrix D has distinct eigenvalues, λ_i. Otherwise the eigenvectors which correspond to the same eigenvalue might be substituted by their linear combinations without affecting the previous equality. So, supposing that D has distinct values in its diagonal, rewriting the equation (4)

$$AR_{s1}A^TE = AR_{s2}A^TED \tag{5}$$

if A is an invertible matrix, we can multiply both sides of the equality by A^{-1} and changing

$$E_s = A^TE \tag{6}$$

The new equality, $R_{s1}E_s = R_{s2}E_sD$, is the eigendecomposition of the source pencil and E_s is its eigenvector matrix. The normalized eigenvectors for a particular eigenvalue are related by $e_s = \alpha A^Te$ where α is a constant that normalizes, to unity the length, the eigenvectors

In what concerns the blind source separation problem the eigenvector matrix E will be an approximation to inverse of mixing matrix, if the E_s is the identity matrix (or a permutation). This is a fact when the matrix pencil of the source signals are both diagonal.

When the mixing matrix is a $m \times n$ $(m > n)$ the equation (5) might written using block matrix notation. Considering A and E divided into two blocks: A into $A_{H,}$ $n \times n$, and $A_L, (m-n) \times n$; E into $E_H, n \times m$ and $E_L, (m-n) \times m$. Therefore, performing matrix block operations the equation (5) can be written as

$$\begin{array}{c} A_HR_{s1}\Phi = A_HR_{s2}\Phi D \\ A_LR_{s1}\Phi = A_LR_{s2}\Phi D \end{array} \tag{7}$$

where $\Phi = A_H^TE_H + A_L^TE_L = A^TE$ is $n \times m$ matrix. The first equation shows that this case also resumes the relation among congruent pencils. Φ is a matrix that also represents the eigenvector matrix of the source matrix pencil having $(m-n)$ columns of zeroes paired with the eigenvalues in D that does not belong to eigenvalue decomposition of (R_{s1}, R_{s2}). Using this direct approach to solve the blind source separation, it is possible to find out the number of sources because after the separation $(m-n)$ zero amplitude signals are obtained. Nevertheless, the solution is also found using a subset of mixed signals (n signals) to compute the mixed matrix pencil.

In resume, the GED approach to Blind Source Separation is feasible if the congruent pencils have distinct eigenvalues and that the source pencil is the identity matrix (or a permutation). In a practical situation the first condition may not be a real restriction nevertheless, Chang discusses a recursive implementation where the signals are separated into disjoint groups that can be further separated computing additional matrix pencils within the a subset of separated signals corresponding to multiple eigenvalues[4].

2.1. Computing the eigendecomposition of symmetric pencils

There are several ways o compute the eigenvalues or the eigenvectors of a matrix pencil, if at least one of the matrices a symmetric positive definite pencil [10]. A very common approach is to reduce the GED statement to the standard form, i.e. , to the eigenvalue decomposition problem.

Consider the problem of computing the eigenvalues and the eigenvectors of the pencil (R_{x1}, R_{x2})

$$R_{x2}E = R_{x1}ED \tag{8}$$

The reduction of the previous equation to the standard form $\hat{C}Z = ZD$, is achieved by solving the eigendecomposition of the matrix R_{x1}. Then, if the matrix is positive definite, $R_{x1} = S\Delta S^T = S\Delta^{1/2}S^T S\Delta^{1/2}S^T = WW$, and considering $Z = WE$,we can write equation (8) as

$$W^{-1}R_{x2}W^{-1}Z = ZD \tag{9}$$

The previous equation is an eigendecomposition statement of a real symmetric matrix $\hat{C} = W^{-1}R_{x2}W^{-1}$ if R_{x2} is also symmetric positive definite. The transformation matrix $(W^{-1} = S\Delta^{-1/2}S^T)$ should be computed with the non-zero eigenvalues and the corresponding eigenvectors. With the eigendecomposition solution of $\hat{C}$, the eigenvalues of pencil (8)are also available, while the eigenvectors are computed solving the equation $E = W^{-1}Z$.

Usually, the matrix R_{x1} is decomposed using Cholesky approach [10], but with the proposed decomposition W^{-1} can be written using spectral factorization what can be an advantage when an iterative algorithm is required as will show in next section. The Cholesky decomposition $(W^{-1} = \Delta^{-1/2}S^T)$ is also used in algorithms like AMUSE, EFOBI [1] and FastICA [9] to achieve the so called data whitening, but instead of performing a linear transformation on a matrix, the transformation is used on the raw data. In AMUSE and EFOBI algorithms, the second step is a standard eigendecomposition of a matrix which can also be written as a product of matrices very similar to those found in equations (9) and (1). The transformation matrix and the mixing matrix can be identified multiplying, on left and on right, a matrix related with the source signals.

2.2. Iterative eigendecomposition algorithm

The strategy to compute the generalized eigenvalue decomposition of two symmetric positive-definite matrices can be resumed as follows:

- Compute the eigenvalue decomposition (S, Δ) of one of the matrices.
- Compute the matrix $W^{-1} = S\Delta^{-1/2}S^T$ and transform the second matrix
- Compute the eigenvalue decomposition of the transformed matrix $\hat{C}$

Usually the solution to eigendecomposion of the pencil is achieved by finding a solution for each step but each one has effectively an iterative solution. The standard eigendecomposition, of the first and third steps, can be achieved with power method and deflation techniques as briefly described in appendix, and the transformation matrix can be written as a spectral factorization

$$W^{-1} = \sum_i \frac{1}{\sqrt{\delta_i}} s_i s_i^T \tag{10}$$

Therefore, the spectral factorization turns the computation of the transformation matrix into an iterative procedure using a criterion to include a pair (s_i, δ_i) of the first eigendecomposition. The criterion can be the mean square error $e_i^T e_i$, where $e_i = R_{x1}s_i - \delta_i s_i$. Then, after having an estimative to the transformation matrix, the eigendecomposition of $\hat{C}$ can start. This matrix is a rank deficient matrix, during the first iterations, since W^{-1} is rank deficient. The number of eigenvalues used to compute the transformation matrix will be the number of eigenvalues/ eigenvectors of matrix pencil.

The application of this strategy to construct a batch (R_{x1} and R_{x2} are computed before starting the eigendecomposition) algorithm can be resumed into the following steps

Given initial estimates:

- to the set of eigenvector $s_m(0) \neq s_{m-1}(0) \cdots \neq s_1(0)$ of the matrix $R_{x1} \in \Re^{mxm}$

- to the set of eigenvectors $z_m(0) \neq z_{m-1}(0) \cdots \neq z_1(0)$ of the transformed matrix $\hat{C}$.

$for\ k = 1, 2...$

% On-line estimation of R_{x1} and R_{x2}

-Estimate (s_i, δ_i) and e_i, $i = 1 \ldots m$ (see appendix)

$W^{-1} = 0$

$for\ j = 1$ to m

$n = 0$

$if\ (\delta_j(k) > 0\ and\ e_j(k) < tol)$

$W^{-1} = W^{-1} + s_j(k)(s_j(k))^T / \sqrt{\delta_j(k)}$

$n = n + 1$

end

-Transform the second atrix $\hat{C} = W^{-1}R_{x2}W^{-1}$

-Estimate the (z_i, λ_i) $i = 1...m$ of matrix $\hat{C}$

end

-λ_i are the n eigenvalues of (R_{x1}, R_{x2}) and the eigenvectors are $W^{-1}z_i$ (which might be normalized to unity length)

-$n = m$ if the matrices are not rank deficient.

end

A GED on-line algorithm requires that the matrices are computed within the iterative procedure, i.e. , the matrices R_{x1} and R_{x2} must be actualized whenever a new sample of each signal is available. Then, at the beginning of the main loop the proper actualization for each matrix must be included.

3. NUMERICAL SIMULATION

The experimental study presented on this work concerns the evaluation of the GED approach, computing the correlation matrices on filtered data, and the performance of the proposed algorithm to achieve the eigendecompostion. The experiences were realized with the FastICA toolbox demo signals[9], using random square and rectangular mixing matrices.

3.1. Separation using GED on filtered signals

In this experimental study both the matrices of the pencil are correlation matrices computed at the input and at the output of a finite impulse response(FIR). Considering a segment of mixed signals Y, an $m \times N$ matrix, the pencil is $(\frac{1}{N}YY^T, \frac{1}{N}YH^THY^T)$, where YH^T is the product of each raw of Y with the filter convolution matrix H [6]. The filter used on this experiment is a two samples mean, then H has only two diagonals with non-zero entries. The computation of the separation matrix was achieved using the *eig (A,B)* command of Matlab.

The performance of this methodology is compared with FastICA performance. The evaluation of the methods is achieved using the performance index parameter. The parameter computes the degree of diagonalization of the product (C) of the separation matrix by the mixing matrix. This parameter is computed for each raw i of C and is defined by

$$p_i = \frac{\max(|C_i|)}{\sum |C_{ij}|}. \tag{11}$$

If the matrix C is a permutation of a diagonal matrix, the absolute maximum of each raw must belong to distinct column. In what concerns the separation of the source signals, the column number where the maximum is found out matches the source signal number. So, in a trial, a particular performance index, p_k is considered valid if the maximum belongs to column j with no maximum of other raw, otherwise the source signal j is not extracted. The mean value of the performance index for each source signal, when a separation is achieved, is computed for the two methods (tables 1 and 2). Both methods separate all the sources in every trial (of a total 100) and the performance index is very close both for square and rectangular mixing matrix.

$Method$	$Sc\,\#1$	$Sc\,\#2$	$Sc\,\#3$	$Sc\,\#4$
GED/FIR	0.98	0.94	0.96	0.90
$FastICA$	0.95	0.95	0.94	0.87

Table 1. Using the signals of FastIca toolbox, with 4x4 mixing matrix -the separation index

$Method$	$Sc\,\#1$	$Sc\,\#2$	$Sc\,\#3$	$Sc\,\#4$
GED/FIR	0.99	0.94	0.96	0.90
$FastICA$	0.96	0.95	0.95	0.87

Table 2. Using the signals of FastIca toolbox, with 6x4 mixing matrix -the separation index

3.2. Algorithm performance

Preliminary results concerning the performance of the algorithm are discussed in this section. The performance was evaluated for batch and on-line implementations. The batch performance of the algorithm is comparable to the eig command of Matlab as we can see in first and second columns of the tables (3 and 4). Figure (1) shows the evolution of the pencil eigenvalues along the first 50 iterations, and it can be verified that the algorithm converges very fast towards the final values. On both plots, it is visible the influence of the transformation matrix on the convergence characteristics, when $W^{-1} \neq 0$ but rank deficient, the eigenvalues start equal to zero and sequentially change to a non-zero value. But it can be seen that the first eigendecomposition starts a convergence (tolerance for mean square error was 0.01) in less than 15 iterations. The first plot shows a non-rank deficient case while the second plot shows a rank deficient case (there are eigenvalues that remain equal to zero).

In the on-line implementation the correlation matrices are computed iteratively, assuming that each sample of mixed signals and filtered signals are available on iteration i. For instance, the correlation matrix of the mixed signals in iteration i is

$$R_{x1}(i) = (1 - 1/i)R_{x1}(i - i) + (1/i)y(:, i)y(:, i)^T, \tag{12}$$

and the correlation of the filtered mixed signal is computed in a similar way, substituting the vector $y(:, i)$ by the vector of filtered signals

The third column of tables (3 and 4) shows the eigenvalues at the end of the segment of 1000 samples and figure 2 shows the evolution, along the data segment, of the eigenvalues of the pencil. We can see that the convergence starts later than in batch implementa ion, but in very short segments the signals are not stationary. It can be also verified that in rank deficient matrices (second plot of figure 2) the converge is not so fast as in the non-rank deficient matrices. It also presents a more instable behavior on the beginning,

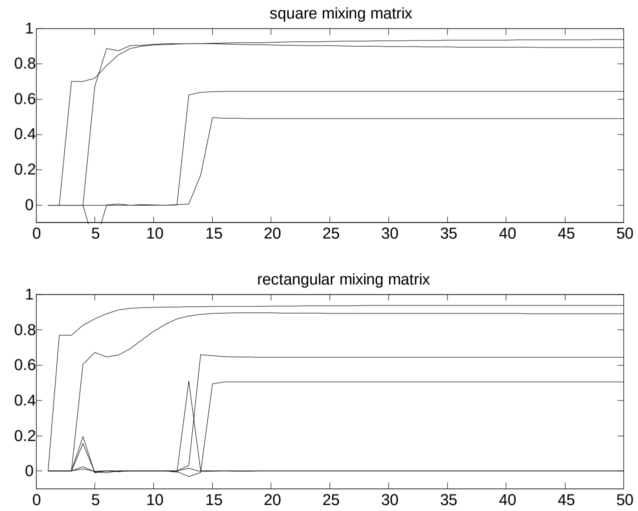

Fig. 1. Batch implementation- evolution of the pencil eigenvalues

Numerical	Batch(1000)	On-line
0.479015705044	0.479015705044	0.478540881424
0.644916582931	0.644916582931	0.644362375553
0.891072116503	0.891072116503	0.893695758628
0.938799917384	0.938799917384	0.937778462743

Table 3. Square mixing matrices: pencil eigenvalues

Numerical	Batch	On-line
-0.004628569602	0.000000000000	0.00000000000
19.19389810755	0.000000000000	0.00000000000
0.506103128425	0.506103128425	0.50621164467
0.644979148253	0.644979148253	0.64389926903
0.891111773836	0.891111773836	0.89357466723
0.938799777695	0.938799777695	0.93758168367

Table 4. Rectangular mixing matrix: pencil eigenvalues

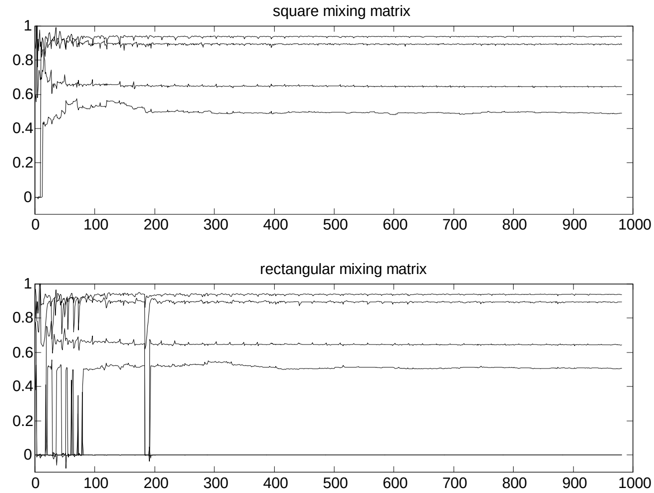

Fig. 2. On-line implementation- evolution of pencil eigenvalues

it was also verified that in the first iterations extremely large eigenvalues can be estimated.

The experiences also show that the eigenvectors are always estimated in descending order of magnitude of the corresponding eigenvalues. This is justified by the application of the power method and deflation techniques applied in the standard eigendecompositions.

In what concerns blind source separation the eigenvector matrices, corresponding to the eigenvalues in the tables, were used to separate the mixed signals. In spite the differences in the eigenvalues, estimated by the on-line and on batch implementation of the algorithm, the performance indexes are very similar (less than 1% for every source signal). Figure 3 shows an example of the separation achieved with the on-line implementation of the algorithm.

4. CONCLUSIONS

The congruent pencil definition, and its extension to pencils with different dimensions, turns the GED formulation to blind source very simple. The method is reliable if equation (6) is applicable, but working with estimates, the source matrix pencil can have very small values off the main diagonal and, its eigenvector matrix is far from being the identity.

The proposed algorithm is efficient and produces results comparable to eig Matlab command, but we are working on the development of other versions that might be less complex. For instance, the mean square error can be avoided as a control parameter if the eigenvalues always present the regular behavior shown in the previous experiences.

One aspect that must be further studied is the influence of noise, both in the GED approach to blind source separation and in the algorithm performance. We think that in cases where the number of mixed signals is higher than the number of source signals the formulation can be easily adapted reformulating the mathematical model to $y(t) = A's'(t)$, where the vector $s'(t)$ includes the noise signals and A' will be the mixing matrix. In what concerns the algorithm, using the correlation matri in the first eigendecomposition, a strategy similar to that used in the whitening phase, the number of source signals can be estimated.

5. APPENDIX

The power method constitutes a systematic and iterative approach to look for vectors in $\Re^m$ which can be the prin-