BLIND SOURCE SEPARATION OF BILINEARLY MIXED SIGNALS

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ABSTRACT

We propose a blind source separation model for statistically independent source signals, when the mixing operator is bilinear. This model is equivalent to a linear model for separation of pairwise multiplications the source signals. We prove that if the source signals are are colored and have distinct autocorrelation functions on a given set P, then we can extract simultaneously the pairwise multiplications of them, using a generalized eigenvalue problem.

1. INTRODUCTION

The interest of blind signal processing, especially, independent component analysis (ICA) has been increased recently, due to its potential applications in many areas, including brain signal processing and other biomedical signal processing, speech enhancement, wireless communication, geophysical data processing, data mining, etc. (see for instance the book [17] and references therein).

The problem of blind source separation (BSS) is formulated as follows: we can observe sensor signals $\mathbf{x}(k) = [x_1(k), \ldots, x_m(k)]^T$ which are described as

$$\mathbf{x}(k) = \mathbf{H}\mathbf{s}(k) + \mathbf{n}(k), \tag{1}$$

where $\mathbf{H}$ is $m \times n$ full-rank unknown mixing matrix, $\mathbf{s}(k) = [s_1(k), \ldots, s_n(k)]^T$ $(n \leq m)$ is a vector of unknown zero mean colored (i.e. with temporal structure) source signals and $\mathbf{n}(k)$ is a vector of additive white noise. Our objective is to estimate the mixing matrix $\mathbf{H}$ and/or source signals simultaneously or sequentially one-by-one assuming that they are uncorrelated (not necessarily statistically independent) but arbitrarily distributed colored (not independent identically distributed), i.e. we assume that sources satisfy the relation: $E\{s_i(k)\, s_i(k-p)\} \neq 0$ at least for some $p = 1, 2, \ldots$ and have different temporal structures.

The use of second statistics approach for blind separation of temporally correlated sources has been developed and analyzed by many researchers, including Amari [2], Molgdey and Schuster [5], Pham and Garat [7], Belouchrani et al. [3], Cichocki, Rutkowski, Barros, and Oh [13], Cichocki, and Thawonmas [14], Choi and Cichocki [12], Pearlmutter and Parra [6], Mueller et al.[10], etc. Moreover, it should be mentioned that recently several researchers have developed a number of efficient algorithms for sequential blind source extraction, especially works of Delfosse and Loubaton [15], Hyvärinen and Oja [16], etc.

In this paper we consider a new model for BSS problem, when the mixing mapping is bilinear. This model is equivalent to a linear model, if the source signals are presented as pairwise multiplications of statistically independent signals. We prove that, using a generalized eigenvalue problem and second order statistics we can extract simultaneously these pairwise multiplications, assuming that they are colored and have different autocorrelation functions on a given set of delays P.

2. BILINEAR MODEL

Consider the following model:

$$\mathbf{x}(t) = \mathbf{H}(\mathbf{s}(t), \mathbf{s}(t)), t = 1, 2, ...$$

where $\mathbf{H} : \mathbb{R}^{2n} \to \mathbb{R}^{n^2}$ is a bilinear operator. Equivalently,

$$x_i(t) = \mathbf{s}^T(t)\mathbf{H}_i\mathbf{s}(t) = \sum_{k,l=1}^{n} h_{i,k,l}s_k(t)s_l(t),$$

where $\mathbf{H}_i, i = 1, ..., n^2$ is a symmetric matrix with elements $h_{i,k,l}$.

Define the matrix $\mathbf{A} \in \mathbb{R}^{n^2 \times n^2}$ with elements $a_{i,j} = h_{i,k,l}$ for $j = n(k-1) + l, i = 1, ..., n^2$ $k, l = 1, ..., n$ and the signals $\tilde{s}_j(t) = s_k(t)\mathbf{s}_l(t)$. Then the above bilinear model is equivalent to the following linear model:

$$\mathbf{x}(t) = \mathbf{A}\tilde{\mathbf{s}}(t), t = 1, 2, ...$$

We assume that the matrix $\mathbf{A}$ is nonsingular.

3. BLIND EXTRACTION OF A MULTIPLICATION OF TWO SIGNALS

Let us firstly assume that we want to extract only one single multiplication of two source signals, say $s_k(t)s_l(t)$ from the available sensor vector $\mathbf{x}(t)$. For this purpose we design a single processing unit described as:

$$y_1(t) = \mathbf{w}_1^T \mathbf{x}(t) = \sum_{j=1}^{n^2} w_{1j} x_j(t),$$

where $\mathbf{w}_1 = [w_{11}, w_{12}, \ldots, w_{1n^2}]^T$.

Define

$$c_{k,l} = \sum_{i=1}^{n^2} w_{1i} h_{i,k,l}.$$

Then

$$y(t) = \sum_{k,l=1}^{n^2} c_{k,l} s_k(t) s_l(t).$$

Our objective is to estimate the optimal values for the vector $\mathbf{w}_1$ in such way that the processing unit extracts successfully the multiplication of two sources.

Observation: if the matrix $\mathbf{C} = \sum_{i=1}^{n} w_i \mathbf{H}_i$ (which is symmetric) has only two nonzero elements, say $c_{k,l}$ and $c_{l,k}$, then $y_1(t) = 2c_{k,l}s_k(t)s_l(t)$ for every $t = 1, 2, \ldots$, i.e. we extract the multiplication of the $k-$th and $l-$th sources.

The idea how to find such $\mathbf{w}_1$ is obtained from the following optimization problems.

(P1) maximize $f(\mathbf{w}) = \mathbf{w}^T \mathbf{R}_x(p) \mathbf{w}$
under constraint $\mathbf{w}^T \mathbf{R}_x(0) \mathbf{w} = 1$,

where $\mathbf{R}_x(p) = E\{\mathbf{x}\mathbf{x}_p^T\}$, $\mathbf{x} = \mathbf{x}(k)$, $\mathbf{x}_p(k) = \mathbf{x}(k-p)$, $p \neq 0$ is time delay and E is the expectation (averaging) operator:

$$E\{\mathbf{x}\mathbf{x}_p^T\} = \lim_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbf{x}(k) \mathbf{x}^T(k-p);$$

and

(P2) maximize

$$\tilde{f}(\mathbf{c}) = \sum_{k,l=1}^{n} \left(2c_{k,l}^2 E\{s_k s_k(t-p)\} E\{s_l s_l(t-p)\} + c_{k,k} c_{l,l}\right)$$

under constraint

$$\sum_{k,l=1}^{n} \left(2c_{k,l}^2 + c_{k,k} c_{l,l}\right) = 1,$$

where the vector $\mathbf{c}$ has components $c_j = c_{k,l}$ for $j = n(k-1) + l$.

Define the matrix $\mathbf{S}(p)$ with elements $s_{i,j}(p) = 1$ for $i = n(k-1)+k, j = n(l-1)+l$ and $s_{j,j}(p) = E\{s_k(t)s_k(t-p)\}E\{s_l(t)s_l(t-p)\}$, $j = n(k-1)+l$, $k, l = 1, \ldots, n$; every other elements of $\mathbf{S}(p)$ are zero by definition.

We have the following representation:

$$E\{\mathbf{x}\mathbf{x}_p^T\} = \mathbf{A}\mathbf{S}(p)\mathbf{A}^T \tag{2}$$

Lemma 1 *The problems (P1) and (P2) are equivalent in sense that*

(a) their optimal values are equal ($f_{max} = \tilde{f}_{max}$), and

(b) $\mathbf{w}^*$ is a solution of (P1) if and only if the vector $\mathbf{c}^*$ with components $c_{k,l}^* := \sum_{i=1}^{n^2} w_i^* h_{i,k,l}$ describing the mixing and extraction procedure is a solution of (P2).

The above lemma follows easily from (2) and is a particular case of the following theorem.

Theorem 1 *Suppose that the source signals $\{s_i(t)\}_{i=1}^n$ are uncorrelated and colored (i.e. $\mathbf{R}_s(p) = E\{\mathbf{s}\mathbf{s}_p^T\}$ $p = 1, 2, \ldots$ are diagonal matrices with at least some nonzero entries (on the main diagonal) and $\mathbf{R}_s(0) = \mathbf{I}$). Then:*

(a) If (P1) has unique solution (up to sign) $\mathbf{w}^*$, then the solution of (P2) is $\mathbf{c}^* = \mathbf{A}^T\mathbf{w}^*$ and this solution has at most two nonzero elements.

(b) Moreover, the set of the generalized eigenvalues of the matrix pencil $(\mathbf{R}_x(p), \mathbf{R}_x(0))$ and the set of the generalized eigenvalues of the matrix pencil $(\mathbf{S}(p), \mathbf{S}(0))$ coincide.

(c) If a generalized eigenvalue λ_i has multiplicity 1, then the corresponding eigenvectors of $(\mathbf{S}(p), \mathbf{S}(0))$ has at most 2 nonzero components.

Proof. (a) The assertion follows by Lemma 1 and assuming the contrary.

b) The assertion follows from representation (2).

(c) The assertion follows from representation (2) and by the special structure of $\mathbf{S}(p)$. ∎

4. SUFFICIENT CONDITION FOR SIMULTANEOUS BLIND SOURCE SEPARATION OF MULTIPLICATIONS OF SIGNALS

We define autocorrelation functions of second order for the source signals by

$$r_i(p) = E\{s_i(k)s_i(k-p)\},$$

where $p \geq 1$ is a time delay. For a given finite subset $P = \{p_1, \ldots, p_L\}$ of natural numbers we introduce the following condition:

$$\forall i, j \neq i \quad \exists l(i,j) \in \{1, \ldots, L\} : r_i(p_{l(i,j)}) \neq r_j(p_{l(i,j)}), \qquad (\mathbf{DAF}(P))$$

i.e. the sources have different autocorrelation functions on the set P, at least for some discrete time delay $p(i,j) \in P$.

Define a covariance matrix of the sensor signals by

$$\mathbf{R}_x(p) = E\{\mathbf{x}\mathbf{x}_p^T\},$$

and similarly, a covariance matrix of the source signals by

$$\mathbf{R}_s(p) = E\{\mathbf{s}\mathbf{s}_p^T\},$$

where $\mathbf{x}_p = \mathbf{x}(k-p), \mathbf{x} = \mathbf{x}(k), \mathbf{s}_p = \mathbf{s}(k-p), \mathbf{s} = \mathbf{s}(k)$.

We recall that the source signals are *uncorrelated*, if $\mathbf{R}_s(p)$ are diagonal matrices for every $p \geq 1$. Note that in this case the diagonal elements of $\mathbf{R}_s(p)$ are $r_i(p), i = 1, ..., n$. If the source signals are statistically independent, then this condition is satisfied, but the converse assertion is not always true. We say that the sources are *colored*, if for some vector $p_0 \geq 1$ the matrix $\mathbf{R}_s(p_0)$ is nonzero (diagonal) matrix.

Let the set P has L elements $\{p_1, ..., p_L\}$. For a given vector $\mathbf{b} \in \mathbb{R}^L$ define

$$\mathbf{X}(\mathbf{b}) = \sum_{i=1}^{L} b_i \mathbf{R}_x(p_i),$$

and similarly for the source signals

$$\mathbf{S}(\mathbf{b}) := \sum_{i=1}^{L} b_i \mathbf{S}(p_i).$$

Theorem 1 *Assume that the source signals are colored and uncorrelated and condition (***DAF***(P)) is satisfied. Then there exists a vector* $\mathbf{b} \in \mathbb{R}^{\mathrm{L}}$ *such that the eigenvalues of the matrix pencil* $(\mathbf{X}(\mathbf{b}), \mathbf{R}_x(0))$ *have multiplicity 1 and* $\mathbf{w}_i^T \mathbf{x}(t) = K s_k(t) s_l(t), i = n(k-1) + l$ *(*K *is a constant). Furthermore, the set* $B(L)$ *of all vectors* $\mathbf{b} \in \mathbb{R}^{\mathrm{L}}$ *with this property form an open subset of* $\mathbb{R}^{\mathrm{L}}$*, whose complement has a Lebesgue measure zero.*

Proof. (a) We have $\mathbf{X}(\mathbf{b}) = \mathbf{A}\mathbf{S}(\mathbf{b})\mathbf{A}^T$, and the matrix $\mathbf{S}(\mathbf{b})$ has at most two non-zero elements in each column and each row. Observe that the matrix pencils $(\mathbf{X}(\mathbf{b}), \mathbf{R}_x(0))$ and $(\mathbf{S}(\mathbf{b}), \mathbf{S}(0))$ have the same eigenvalues. It is easy to see that the complement of $B(L)$ is a finite union of subspaces of $\mathbb{R}^L$. If we prove that $B(L)$ is nonempty, then every of these subspaces must be proper (i.e. different from $\mathbb{R}^L$), consequently, with a Lebesgue measure zero (with respect to $\mathbb{R}^L$), therefore the complement of $B(L)$ must have a Lebesgue measure zero too.

The proof that $B(L)$ is non-empty is similar to the proof that analogous set is non-empty (see [4]). ∎

Remark In practical situations, when the sources are supposed to be very different (i.e. to have different autocorrelation functions for almost all delays p), the set P can be chosen to consist of only one element and to take trials for different p, until obtaining distinct eigenvalues of the matrices $\mathbf{X}(\mathbf{b})$.

5. CONCLUSION

We propose a bilinear model for the Blind Source Separation problem and show that it is equivalent to a linear BSS problem for source signals consisting of pairwise multiplications of the original sources. We give conditions under which it is possible extraction of such pairwise multiplications and describe a procedure based on generalized eigenvalue problem for this purpose.

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