When tensor decomposition meets compressed sensing

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Blind Source Separation & Sparse Representation

$$\mathbf{x} = \mathbf{H}\,\mathbf{s}$$

$\mathbf{H}$ is $K \times P$, *underdetermined*: $K < P$

- *Sparse representation*: Columns $\mathbf{h}_n \in \mathbb{D}$, known dictionary
- *BSS*: $\mathbf{H}$ unknown
  - $\mathbf{s}$ sparse
  - $\mathbf{s}$ not sparse

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Blind identification of linear mixtures

- **Linear mixtures:**

$$\mathbf{x} = \mathbf{H}\,\mathbf{s}$$

If s_ℓ *statistically independent*, we have the *core equation*:

$$\Psi(\mathbf{u}) = \sum_{\ell=1}^{P} \varphi_\ell(\mathbf{u}^{\mathsf{T}}\mathbf{H})$$

- Take 3rd derivatives at point $\mathbf{u}$:

$$T_{ijk}(\mathbf{u}) = \sum_{\ell=1}^{P} H_{i\ell}\, H_{j\ell}\, H_{k\ell}\;\; C_{\ell\ell\ell}(\mathbf{u}) \tag{1}$$

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At $\mathbf{u} = \mathbf{0}$ ➠ symmetric decomposition of T_{ijk} *[HOS]*
At $\mathbf{u} \neq \mathbf{0}$ ➠ *[Taleb, Comon-Rajih, Yeredor]*

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Tensors: General items

Arrays and Multi-linearity

- A tensor of *order d* is a multi-linear map:

$$\mathcal{S}_1^* \otimes \ldots \otimes \mathcal{S}_m^* \longrightarrow \mathcal{S}_{m+1} \otimes \ldots \otimes \mathcal{S}_d$$

- Once bases of spaces $\mathcal{S}_\ell$ are fixed, they can be represented by *d-way arrays* of coordinates

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Multi-linearity

Compact notation

Linear change in contravariant spaces:

$$T'_{ijk} = \sum_{npq} A_{in} B_{jp} C_{kq} T_{npq}$$

Denoted compactly

$$\mathcal{T}' = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{T} \tag{2}$$

Example: covariance matrix

$$\mathbf{z} = \mathbf{A}\mathbf{x} \Rightarrow \mathbf{R}_z = (\mathbf{A}, \mathbf{A}) \cdot \mathbf{R}_x = \mathbf{A}\,\mathbf{R}_x \mathbf{A}^{\mathsf{T}}$$

Decomposable tensor

- A dth order "*decomposable*" tensor is the tensor product of d vectors:

$$\mathcal{T} = \mathbf{u} \otimes \mathbf{v} \otimes \ldots \otimes \mathbf{w}$$

and has coordinates $T_{ij\ldots k} = u_i \, v_j \ldots w_k$.

- may be seen as a discretization of multivariate *functions whose variables separate*:

$$t(x, y, \ldots, z) = u(x) \, v(y) \, \ldots \, w(z)$$

- Nothing else but *rank-1* tensors, with forthcoming definition

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Example

- Take $$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
- Then $$\mathbf{v}^{\otimes 3} \stackrel{\text{def}}{=} \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} = \left[\begin{array}{rr|rr} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{array} \right]$$
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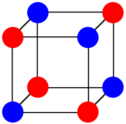

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From SVD to tensor decompositions

Matrix SVD, $\mathbf{M} = (\mathbf{U}, \mathbf{V}) \cdot \mathbf{\Sigma}$, may be extended in at least two ways to tensors

- **Keep orthogonality:** Orthogonal Tucker, HOSVD

$$\mathcal{T} = (\mathbf{U}, \mathbf{V}, \mathbf{W}) \cdot \mathcal{C}$$

$\mathcal{C}$ is $R_1 \times R_2 \times R_3$: *multilinear rank* $= (R_1, R_2, R_3)$

- **Keep diagonality:** Canonical Polyadic decomposition (CP)

$$\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$$

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Exact Canonical Polyadic (CP) decomposition

Canonical Polyadic (CP) decomposition

- Any $I \times J \times \cdots \times K$ tensor $\mathcal{T}$ can be decomposed as

$$\mathcal{T} = \sum_q \lambda_q \, \mathbf{u}^{(q)} \otimes \mathbf{v}^{(q)} \otimes \ldots \otimes \mathbf{w}^{(q)}$$

➽ "Polyadic form" *[Hitchcock'27]*

- The *tensor rank* of $\mathcal{T}$ is the minimal number $R(\mathcal{T})$ of "decomposable" terms such that equality holds.
- May impose unit norm vectors $\mathbf{u}^{(q)}$, $\mathbf{v}^{(q)}$, $\ldots$ $\mathbf{w}^{(q)}$

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[Courtesy of L-H.Lim]

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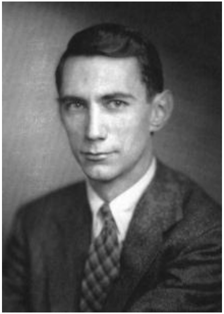


Claude Elwood Shannon (1916-2001)

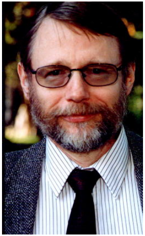
Towards a unique terminology

- Minimal *Polyadic Form [Hitchcock'27]*
- *Canonical decomposition [Weinstein'84, Carroll'70, Chiantini-Ciliberto'06, Comon'00, Khoromskij, Tyrtyshnikov]*
- *Parafac [Harshman'70, Sidiropoulos'00]*
- *Optimal computation [Strassen'83]*
- Minimum-length *additive decomposition* (AD) *[Iarrobino'96]*

Suggestion:

- *Canonical Polyadic decomposition (CP) [Comon'08, Grasedyk, Espig...]*
- *CP* does also already stand for Candecomp/Parafac *[Bro'97, Kiers'98, tenBerge'04...]*

Psychometrics



Richard A. Harshman
(1970)

J. Douglas Carroll
(1970)

Uniqueness: Kruskal 1/2

The *Kruskal rank* of a matrix $\mathbf{A}$ is the maximum number k_A, such that *any* subset of k_A columns are linearly independent.

Uniqueness: Kruskal 2/2

Sufficient condition for uniqueness of CP
[Kruskal'77, Sidiropoulos-Bro'00, Landsberg'09]:

Essential uniqueness is ensured if tensor rank R is below the so-called *Kruskal's bound*:

$$2R + 2 \leq k_A + k_B + k_C \tag{3}$$

or *generically*, for a tensor of order d and dimensions N_ℓ:

$$2R \leq \sum_{\ell=1}^{d} \min(N_\ell, R) - d + 1$$

- Bound much smaller than expected rank:
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Rank-3 example 1/2

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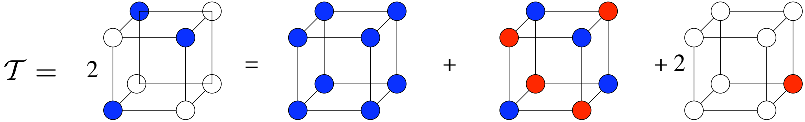

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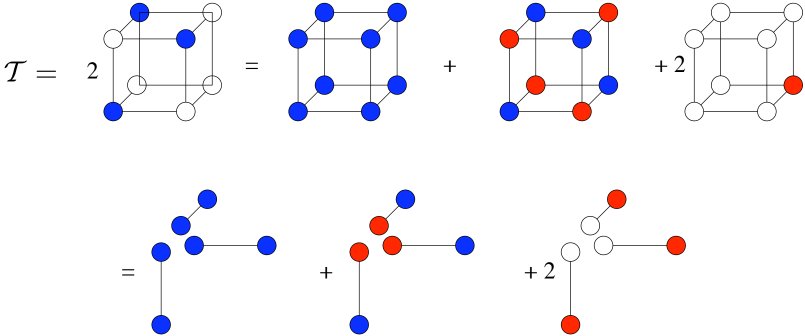

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Rank-3 example 2/2

Conclusion: the $2 \times 2 \times 2$ tensor

$$\mathcal{T} = 2 \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right]$$

admits the CP

$$\mathcal{T} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}^{\otimes 3} + 2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}^{\otimes 3}$$

and has rank 3, hence *larger* than dimension

Approximate Canonical Polyadic (CP) decomposition

Why need for approximation?

- Additive noise in measurements
- Noise has a continuous probability distribution
- Then tensor rank is *generic*
- Hence there are often *infinitely many* CP decompositions

➠ Approximations aim at getting rid of noise, *and* at restoring uniqueness:

$$Arg \inf_{\mathbf{a}(p),\mathbf{b}(p),\mathbf{c}(p)} ||\mathcal{T} - \sum_{p=1}^{r} \mathbf{a}(p) \otimes \mathbf{b}(p) \ldots \otimes \mathbf{c}(p)||^2 \quad (4)$$

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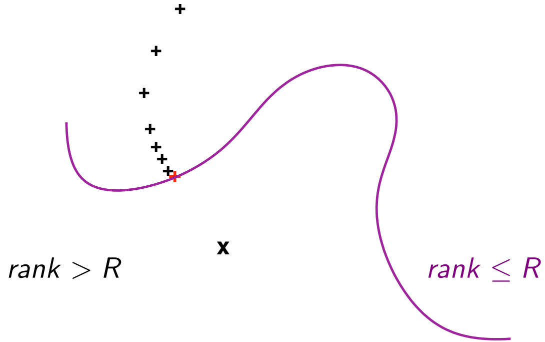
Border rank

$\mathcal{T}$ has *border rank* R iff it is the limit of tensors of rank R, and not the limit of tensors of lower rank.
[Bini'79, Schönhage'81, Strassen'83, Likteig'85]



Example

Let $\mathbf{u}$ and $\mathbf{v}$ be non collinear vectors. Define $\mathcal{T}_0$ *[Comon et al.'08]*:

$$\mathcal{T}_0 = \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$$

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Then $\mathcal{T}_\varepsilon \to \mathcal{T}_0$ as $\varepsilon \to 0$

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Ill-posedness

Tensors for which $\underline{\text{rank}}\{\mathcal{T}\} < \text{rank}\{\mathcal{T}\}$ are such that the approximating sequence contains several decomposable tensors which

- tend to infinity *and*
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Remedies

1. Impose *orthogonality* of columns within factor matrices *[Comon'92]*
2. Impose *orthogonality* between decomposable tensors *[Kolda'01]*
3. Prevent tendency to infinity by *norm constraint* on factor matrices *[Paatero'00]*
4. Nonnegative tensors: impose decomposable tensors to be *nonnegative* *[Lim-Comon'09]* $\to$ "nonnegative rank"
5. Impose *minimal angle* between columns of factor matrices *[Lim-Comon'10]*

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2. Impose *orthogonality* between decomposable tensors *[Kolda'01]*
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4. Nonnegative tensors: impose decomposable tensors to be *nonnegative* *[Lim-Comon'09]* $\rightarrow$ "nonnegative rank"
5. Impose *minimal angle* between columns of factor matrices *[Lim-Comon'10]*

Nonnegativity constraint: example

Fluorescence Spectroscopy 1/3

An optical excitation produces several effects

- Rayleigh diffusion
- Raman diffusion
- Fluorescence

  At low concentrations, Beer-Lambert law can be linearized *[Luciani'09]*

$$I(\lambda_f, \lambda_e, k) = I_o \sum_{\ell} \gamma_\ell(\lambda_f)\, \epsilon_\ell(\lambda_e)\, c_{k,\ell}$$

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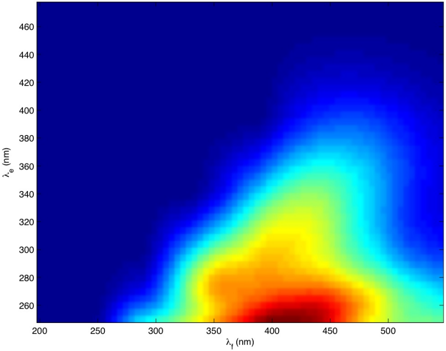
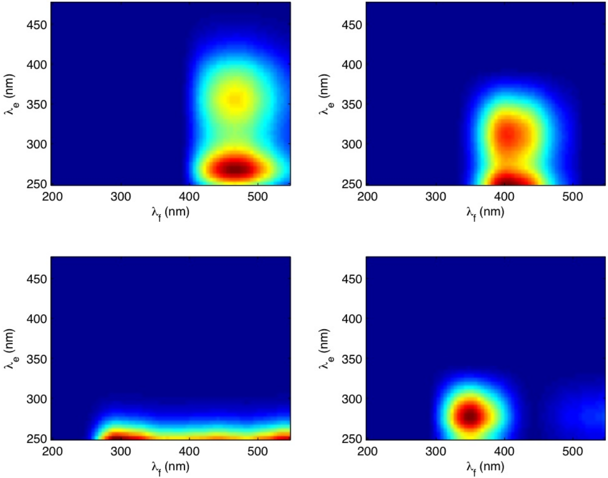
Fluorescence Spectroscopy 2/3

Mixture of 4 solutes (one concentration shown)



Fluorescence Spectroscopy 3/3

Obtained results with $R = 4$



Approximate CP decomposition: Coherence constraint

Coherence

- **Definition:** *[Donoho'03, Gribonval'03, Candès'07]*
  let $\mathbf{A}$ a matrix with unit-norm columns, $\mathbf{a}_p$.

$$\mu_A = \max_{p \neq q} |\langle \mathbf{a}_p, \mathbf{a}_q \rangle| \tag{5}$$

- this *"coherence of* $\mathbf{A}$*"* is used in Sparse Representation theory

Existence

Best rank-R approximate under angular constraint

- **Proposition:** *[Lim-Comon'2010]*
  Let $\mathcal{L}$ diagonal, and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have R unit norm columns.
  If $R < [\mu_A \mu_B \mu_C]^{-1}$, then:

$$\inf_{\mathbf{A},\mathbf{B},\mathbf{C}} ||\mathcal{T} - (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}||$$

is attained.

Uniqueness: Back to Kruskal

- **Lemma:** (spark) *[Gribonval'03]*

$$k_A \geq \frac{1}{\mu_A}$$

Uniqueness

- **Proposition:** *[Lim-Comon'10]*
  If $\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$, with $\lambda_p \neq 0$ for $1 \leq p \leq R$, **A**, **B**, **C** matrices with unit norm columns, and:

$$2R + 2 \leq \frac{1}{\mu_A} + \frac{1}{\mu_B} + \frac{1}{\mu_C} \tag{6}$$

  then $\mathcal{T}$ has *a unique CP* decomposition of rank R, up to unit modulus scalar factors (ρ_A, ρ_B, ρ_C), $\rho_A \rho_B \rho_C = 1$.

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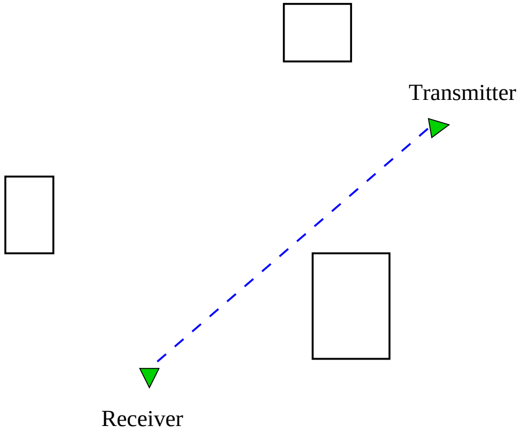
Angular constraint: example

Antenna Array Processing



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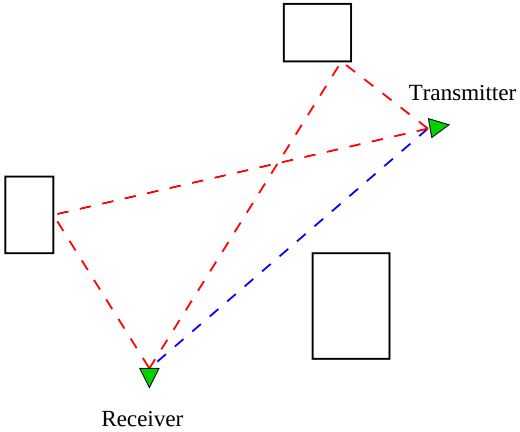

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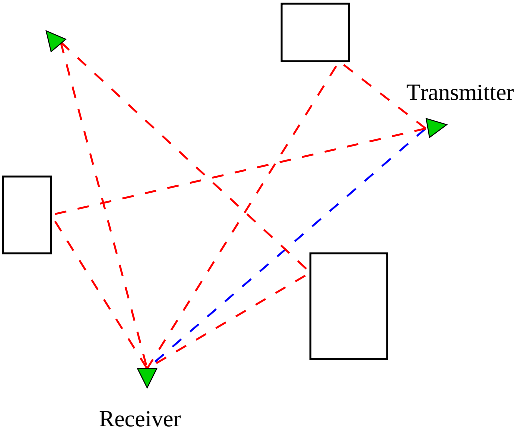

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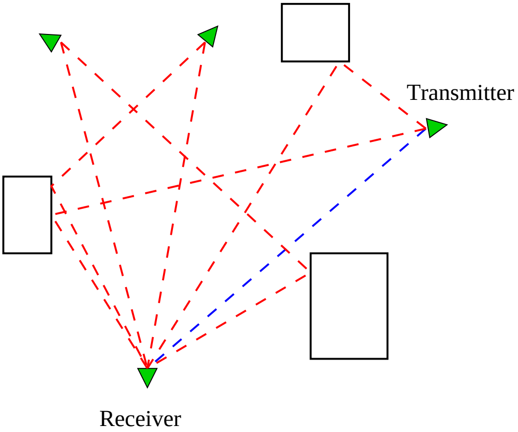

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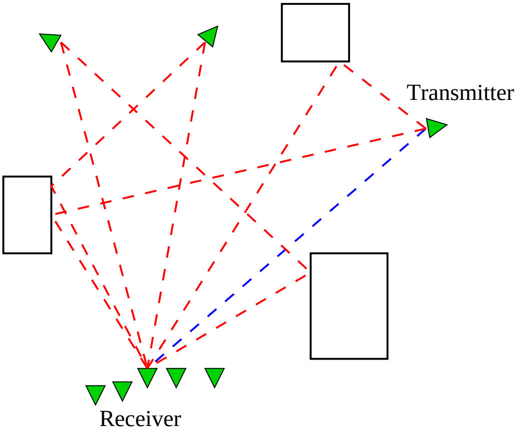

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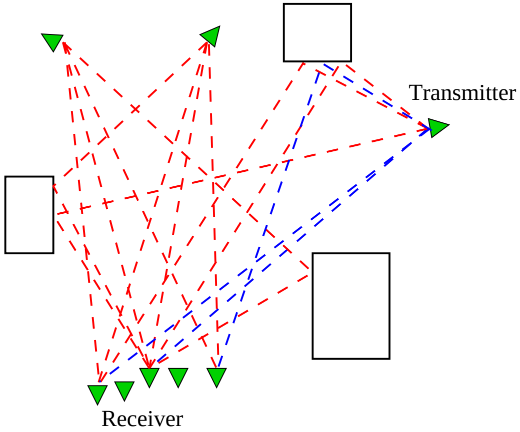

Narrow band model in the far field

Modeling the signals received on an array of antennas generally leads to a *matrix decomposition*:

$$T_{ij} = \sum_q a_{iq}\, s_{jq}$$

i: space	q: path, source	**A**: antenna gains
j: time		**S**: transmitted signals

But in the presence of additional *diversity*, a tensor can be constructed, thanks to new index p

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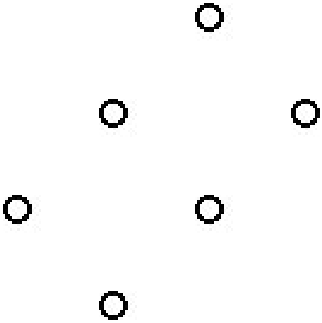
Possible diversities in Signal Processing

- space
- time
- space translation (array geometry)
- time translation (chip)
- frequency/wavenumber (nonstationarity)
- excess bandwidth (oversampling)
- cyclostationarity
- polarization
- finite alphabet
- ...

Space translation diversity (1/2)



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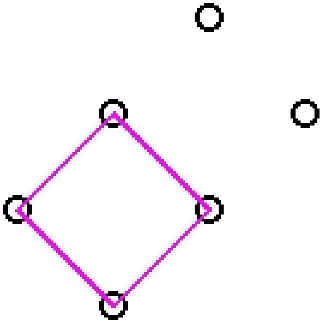

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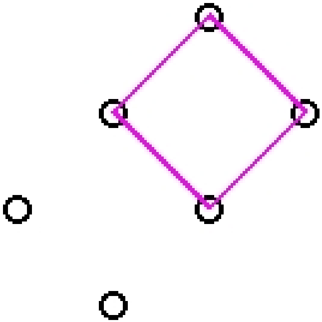

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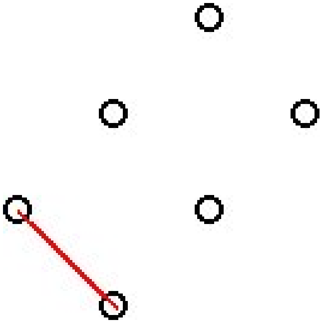

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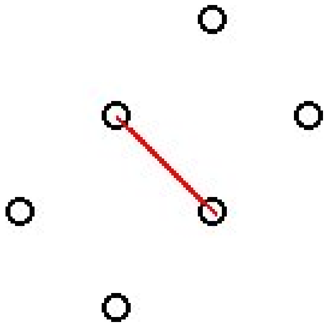

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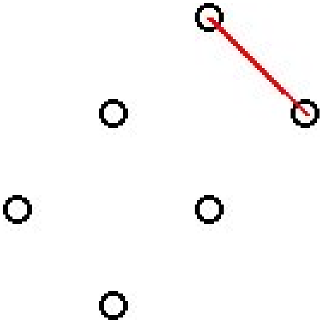

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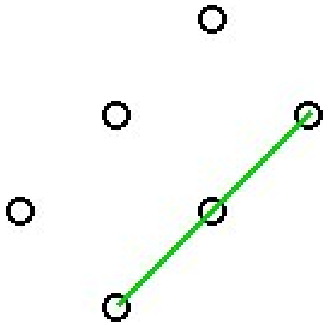

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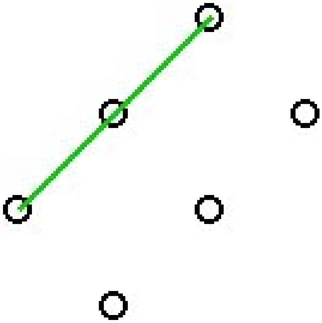

Space translation diversity (2/2)

A_{iq}: gain between sensor i and source q
H_{pq}: transfer between reference and subarray p
S_{jq}: sample j of source q
β_q: path loss, $\mathbf{d}_q$: DOA, $\mathbf{b}_i$: sensor location

Tensor model (NB far field) *[Sidiropoulos'00]*

- Reference subarray: $A_{iq} = \beta_q \exp(\jmath \frac{\omega}{C} \mathbf{b}_i^\mathsf{T} \mathbf{d}_q)$
- Space translation (from reference subarray):
  $\beta_q \exp(\frac{\omega}{C} [\mathbf{b}_i + \mathbf{\Delta}_p]^\mathsf{T} \mathbf{d}_q) \stackrel{\text{def}}{=} A_{iq} H_{pq}$
- Trilinear model:

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Meaning of coherence

1. Unconstrained joint source estimation & localization *[Sidiropoulos'2000]* (ill-posed if approximate)
2. Coherence-constrained joint source estimation & localization *[Lim-Comon'2010]*

   - *time diversity:* $\mu_A \rightarrow$ *cross correlation*
   - *space diversity:* μ_B, $\mu_C \rightarrow$ *angular separation*

Unadressed topics

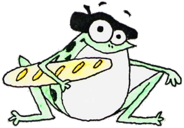
- spread spectrum communications
- brain inverse problems
- medical imaging (MRI)
- NL filtering
- noise reduction
- compression (Tensor trains...)
- probability
- hyperspectral imaging
- structured tensors
- convolutive mixtures
- nonnegative factors
- ...
- Algorithms

QUIZ: Why use tensors?

Main reason: essential uniqueness
- Identifiability recovery, *up to scale-permutation*

Sometimes: powerful *deterministic* approaches

Secondary reason: more sources with fewer sensors
- Matrices **A**, **B**, **C** may have more columns than rows



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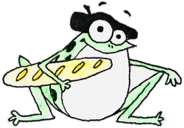

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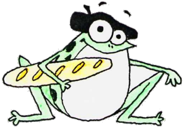

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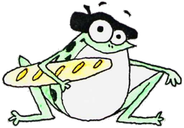

Other perspectives

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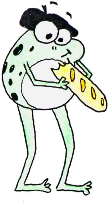

Only when the last tree has died, the last river has been poisoned and the last fish has been caught will we realize that we cannot eat money.
Cree proverb