Intro 00 Exact CP 00000000 Approx CP

NonNeg 000 Coherence

END

AAP

When tensor decomposition meets compressed sensing

Pierre Comon

I3S, CNRS, University of Nice - Sophia Antipolis, France Collaborator: Lek-Heng Lim

Sept. 27-30, 2010

Pierre Comon

LVA/ICA - Sept. 2010

・ロト ・聞 ト ・ヨト ・ヨトー

1 / 43

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
•0	00000	00000000	00000	000	0000	000000	000
Sparse repres	entation & BSS						

$\mathbf{x} = \mathbf{H}\,\mathbf{s}$

H is $K \times P$, underdetermined: K < P

Sparse representation: Columns h_n ∈ D, known dictionary
 BSS: H unknown

S sparse

s not sparse

Pierre Comon

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
•0	00000	0000000	00000	000	0000	000000	000
Sparse represe	ntation & BSS						

$\mathbf{x} = \mathbf{H} \, \mathbf{s}$

H is $K \times P$, underdetermined: K < P

Sparse representation: Columns h_n ∈ D, known dictionary
 BSS: H unknown

s sparse

s not sparse

LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
•0	00000	0000000	00000	000	0000	000000	000
Sparse represe	ntation & BSS						

$\mathbf{x} = \mathbf{H} \, \mathbf{s}$

H is $K \times P$, underdetermined: K < P

Sparse representation: Columns h_n ∈ D, known dictionary
 BSS: H unknown

• s sparse

• s not sparse

LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
•0	00000	0000000	00000	000	0000	000000	000
Sparse represe	ntation & BSS						

$\mathbf{x} = \mathbf{H} \, \mathbf{s}$

H is $K \times P$, underdetermined: K < P

Sparse representation: Columns h_n ∈ D, known dictionary
 BSS: H unknown

s sparse

s not sparse

LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
•0	00000	0000000	00000	000	0000	000000	000
Sparse represe	ntation & BSS						

$\mathbf{x} = \mathbf{H} \, \mathbf{s}$

H is $K \times P$, underdetermined: K < P

Sparse representation: Columns h_n ∈ D, known dictionary
 BSS: H unknown

• s sparse

• s not sparse

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
0.	00000	00000000	00000	000	0000	000000	000
Statistical ap	proach						

Linear mixtures:

$\mathbf{x} = \mathbf{H}\,\mathbf{s}$

If s_{ℓ} statistically independent, we have the core equation:

$$\Psi(\mathbf{u}) = \sum_{\ell=1}^{P} \varphi_{\ell}(\mathbf{u}^{\mathsf{T}} \mathbf{H})$$

Take 3rd derivatives at point u:

$$T_{ijk}(\mathbf{u}) = \sum_{\ell=1}^{P} H_{i\ell} H_{j\ell} H_{k\ell} C_{\ell\ell\ell}(\mathbf{u})$$
(1)

< ロ > < 個 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ >

Pierre Comon

LVA/ICA - Sept. 2010

3 / 43

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
0●	00000	00000000	00000	000	0000	000000	000
Statistical ap	proach						

Linear mixtures:

 $\mathbf{x} = \mathbf{H}\,\mathbf{s}$

If s_{ℓ} statistically independent, we have the core equation:

$$\Psi(\mathbf{u}) = \sum_{\ell=1}^{P} \varphi_{\ell}(\mathbf{u}^{\mathsf{T}} \mathbf{H})$$

Take 3rd derivatives at point u:

$$T_{ijk}(\mathbf{u}) = \sum_{\ell=1}^{P} H_{i\ell} H_{j\ell} H_{k\ell} C_{\ell\ell\ell}(\mathbf{u})$$
(1)

At $u = 0 \implies$ symmetric decomposition of T_{ijk} (HOS) At $u = 0 \implies (1.5)$ (Labor Composition for (1.5) (1

Pierre Comon

LVA/ICA - Sept. 2010

3 / 43

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
0●	00000	00000000	00000	000	0000	000000	000
Statistical ap	proach						

Linear mixtures:

 $\mathbf{x} = \mathbf{H}\,\mathbf{s}$

If s_{ℓ} statistically independent, we have the core equation:

$$\Psi(\mathbf{u}) = \sum_{\ell=1}^{P} \varphi_{\ell}(\mathbf{u}^{\mathsf{T}} \mathbf{H})$$

Take 3rd derivatives at point u:

$$T_{ijk}(\mathbf{u}) = \sum_{\ell=1}^{P} H_{i\ell} H_{j\ell} H_{k\ell} C_{\ell\ell\ell}(\mathbf{u})$$
(1)

At $\mathbf{u} = \mathbf{0}$ >> symmetric decomposition of T_{ijk} [HOS] At $\mathbf{u} \neq \mathbf{0}$ >> [Taleb, Comon-Rajih, Yeredor]

Pierre Comon

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
0●	00000	00000000	00000	000	0000	000000	000
Statistical ap	proach						

Linear mixtures:

 $\mathbf{x} = \mathbf{H}\,\mathbf{s}$

If s_{ℓ} statistically independent, we have the core equation:

$$\Psi(\mathbf{u}) = \sum_{\ell=1}^{P} \varphi_{\ell}(\mathbf{u}^{\mathsf{T}} \mathbf{H})$$

Take 3rd derivatives at point u:

$$T_{ijk}(\mathbf{u}) = \sum_{\ell=1}^{P} H_{i\ell} H_{j\ell} H_{k\ell} C_{\ell\ell\ell}(\mathbf{u})$$
(1)

At $\mathbf{u} = \mathbf{0} \implies$ symmetric decomposition of T_{ijk} [HOS] At $\mathbf{u} \neq \mathbf{0} \implies$ [Taleb, Comon-Rajih, Yeredor]

Pierre Comon

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
0●	00000	00000000	00000	000	0000	000000	000
Statistical ap	proach						

Linear mixtures:

 $\mathbf{x} = \mathbf{H}\,\mathbf{s}$

If s_{ℓ} statistically independent, we have the core equation:

$$\Psi(\mathbf{u}) = \sum_{\ell=1}^{P} \varphi_{\ell}(\mathbf{u}^{\mathsf{T}} \mathbf{H})$$

Take 3rd derivatives at point u:

$$T_{ijk}(\mathbf{u}) = \sum_{\ell=1}^{P} H_{i\ell} H_{j\ell} H_{k\ell} C_{\ell\ell\ell}(\mathbf{u})$$
(1)

At $\mathbf{u} = \mathbf{0}$ \implies symmetric decomposition of T_{ijk} [HOS] At $\mathbf{u} \neq \mathbf{0}$ \implies [Taleb, Comon-Rajih, Yeredor]

Pierre Comon

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000

Tensors: General items

Pierre Comon

LVA/ICA - Sept. 2010

4 / 43

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	0000	00000000	00000	000	0000	000000	000
Notation							

Arrays and Multi-linearity

• A tensor of *order d* is a multi-linear map:

$$\mathcal{S}_1^* \otimes \ldots \otimes \mathcal{S}_m^* \to \mathcal{S}_{m+1} \otimes \ldots \otimes \mathcal{S}_d$$

- Once bases of spaces S_ℓ are fixed, they can be represented by *d-way arrays* of coordinates
- bilinear form, or linear operator: represented by a matrix
 trilinear form, or bilinear operator: by a 3rd order tensor.

LVA/ICA - Sept. 2010

・ロト ・聞ト ・ヨト ・ヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	0000	00000000	00000	000	0000	000000	000
Notation							

Arrays and Multi-linearity

• A tensor of *order d* is a multi-linear map:

$$\mathcal{S}_1^* \otimes \ldots \otimes \mathcal{S}_m^* \to \mathcal{S}_{m+1} \otimes \ldots \otimes \mathcal{S}_d$$

- Once bases of spaces S_ℓ are fixed, they can be represented by *d-way arrays* of coordinates
- bilinear form, or linear operator: represented by a *matrix* trilinear form, or bilinear operator: by a *3rd order tensor*.

イロト 不得下 イヨト イヨト 二日

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Notation							

Multi-linearity

Compact notation

Linear change in contravariant spaces:

$$T'_{ijk} = \sum_{npq} A_{in} B_{jp} C_{kq} T_{npq}$$

Denoted compactly

$$\mathcal{T}' = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{T}$$
(2)

イロト 不得下 イヨト イヨト 二日

Example: covariance matrix

$$\mathbf{z} = \mathbf{A}\mathbf{x} \Rightarrow \mathbf{R}_z = (\mathbf{A}, \mathbf{A}) \cdot \mathbf{R}_x = \mathbf{A} \, \mathbf{R}_x \mathbf{A}^\mathsf{T}$$

Pierre Comon

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Definitions							

Decomposable tensor

A dth order "decomposable" tensor is the tensor product of d vectors:

$$\mathcal{T} = \mathbf{u} \otimes \mathbf{v} \otimes \ldots \otimes \mathbf{w}$$

and has coordinates $T_{ij...k} = u_i v_j \dots w_k$.

may be seen as a discretization of multivariate *functions* whose variables separate:

$$t(x, y, \dots, z) = u(x) v(y) \dots w(z)$$

Nothing else but rank-1 tensors, with forthcoming definition

LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト 二日

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Definitions							

Decomposable tensor

A dth order "decomposable" tensor is the tensor product of d vectors:

$$\mathcal{T} = \mathbf{u} \otimes \mathbf{v} \otimes \ldots \otimes \mathbf{w}$$

and has coordinates $T_{ij...k} = u_i v_j \dots w_k$.

may be seen as a discretization of multivariate *functions* whose variables separate:

$$t(x, y, \ldots, z) = u(x) v(y) \ldots w(z)$$

Nothing else but rank-1 tensors, with forthcoming definition

LVA/ICA - Sept. 2010

<ロト < 回 > < 回 > < 回 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < 三 > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Definitions							

Decomposable tensor

A dth order "decomposable" tensor is the tensor product of d vectors:

$$\mathcal{T} = \mathbf{u} \otimes \mathbf{v} \otimes \ldots \otimes \mathbf{w}$$

and has coordinates $T_{ij...k} = u_i v_j \dots w_k$.

may be seen as a discretization of multivariate *functions* whose variables separate:

$$t(x, y, \ldots, z) = u(x) v(y) \ldots w(z)$$

■ Nothing else but *rank-1* tensors, with forthcoming definition

LVA/ICA - Sept. 2010

イロト イヨト イヨト イヨト ヨー シタの

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Definitions							
			_	1			
			Exam	Ible			

Take
$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then $\mathbf{v}^{\otimes 3} \stackrel{\text{def}}{=} \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} = \begin{bmatrix} 1 & -1 & | & -1 & 1 \\ -1 & 1 & | & 1 & -1 \end{bmatrix}$

This is a "decomposable" symmetric tensor \rightarrow rank-1



LVA/ICA - Sept. 2010

◆□▶ ◆圖▶ ◆臣▶ ◆臣▶ 三臣

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END				
Definitions	5	00000000	00000	000	0000	000000	000				
			_								
	Example										

Take
$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then $\mathbf{v}^{\otimes 3} \stackrel{\text{def}}{=} \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} = \begin{bmatrix} 1 & -1 & | & -1 & 1 \\ -1 & 1 & | & 1 & -1 \end{bmatrix}$

■ This is a "decomposable" symmetric tensor → rank-1



LVA/ICA - Sept. 2010

◆□▶ ◆圖▶ ◆臣▶ ◆臣▶ 三臣

Intro 00 Definitions	General ○○○●○	Exact CP 00000000	Approx CP 00000	NonNeg 000	Coherence 0000	AAP 000000	END 000
			Evam	nlo			



Take
$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then $\mathbf{v}^{\otimes 3} \stackrel{\text{def}}{=} \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} = \begin{bmatrix} 1 & -1 & | & -1 & 1 \\ -1 & 1 & | & 1 & -1 \end{bmatrix}$

• This is a "decomposable" symmetric tensor \rightarrow *rank-1*



LVA/ICA - Sept. 2010



Matrix SVD, $\boldsymbol{\mathsf{M}}=(\boldsymbol{\mathsf{U}},\boldsymbol{\mathsf{V}})\cdot\boldsymbol{\Sigma},$ may be extended in at least two ways to tensors

■ Keep orthogonality: Orthogonal Tucker, HOSVD

 $\mathcal{T} = (\mathbf{U}, \mathbf{V}, \mathbf{W}) \cdot \mathcal{C}$

C is R₁ × R₂ × R₃: multilinear rank = (R₁, R₂, R₃)
■ Keep diagonality: Canonical Polyadic decomposition (CP)

 $\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$

 \mathcal{L} is $R \times R \times R$ diagonal, $\lambda_i \neq 0$: rank = R.

LVA/ICA - Sept. 2010

イロト 不得下 イヨト イヨト



Matrix SVD, $\mathbf{M} = (\mathbf{U}, \mathbf{V}) \cdot \mathbf{\Sigma}$, may be extended in at least two ways to tensors

■ Keep orthogonality: Orthogonal Tucker, HOSVD

 $\mathcal{T} = (\boldsymbol{\mathsf{U}},\boldsymbol{\mathsf{V}},\boldsymbol{\mathsf{W}})\cdot\mathcal{C}$

C is R₁ × R₂ × R₃: multilinear rank = (R₁, R₂, R₃)
■ Keep diagonality: Canonical Polyadic decomposition (CP)

 $\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$

 \mathcal{L} is $R \times R \times R$ diagonal, $\lambda_i \neq 0$: rank = R.

LVA/ICA - Sept. 2010

イロト 不得下 イヨト イヨト 二日



Matrix SVD, $\boldsymbol{\mathsf{M}}=(\boldsymbol{\mathsf{U}},\boldsymbol{\mathsf{V}})\cdot\boldsymbol{\Sigma},$ may be extended in at least two ways to tensors

■ Keep orthogonality: Orthogonal Tucker, HOSVD

 $\mathcal{T} = (\boldsymbol{\mathsf{U}}, \boldsymbol{\mathsf{V}}, \boldsymbol{\mathsf{W}}) \cdot \mathcal{C}$

C is $R_1 \times R_2 \times R_3$: multilinear rank = (R_1, R_2, R_3) • Keep diagonality: Canonical Polyadic decomposition (CP)

 $\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$

 \mathcal{L} is $R \times R \times R$ diagonal, $\lambda_i \neq 0$: rank = R.

LVA/ICA - Sept. 2010

イロト 不得下 イヨト イヨト 二日



Matrix SVD, $\boldsymbol{\mathsf{M}}=(\boldsymbol{\mathsf{U}},\boldsymbol{\mathsf{V}})\cdot\boldsymbol{\Sigma},$ may be extended in at least two ways to tensors

■ Keep orthogonality: Orthogonal Tucker, HOSVD

 $\mathcal{T} = (\boldsymbol{\mathsf{U}}, \boldsymbol{\mathsf{V}}, \boldsymbol{\mathsf{W}}) \cdot \mathcal{C}$

C is $R_1 \times R_2 \times R_3$: multilinear rank = (R_1, R_2, R_3)

• Keep diagonality: Canonical Polyadic decomposition (CP)

 $\mathcal{T} = (\textbf{A}, \textbf{B}, \textbf{C}) \cdot \mathcal{L}$

 \mathcal{L} is $R \times R \times R$ diagonal, $\lambda_i \neq 0$: rank = R.

LVA/ICA - Sept. 2010

イロト 不得 トイヨト イヨト 二日



Matrix SVD, $\boldsymbol{\mathsf{M}}=(\boldsymbol{\mathsf{U}},\boldsymbol{\mathsf{V}})\cdot\boldsymbol{\Sigma},$ may be extended in at least two ways to tensors

■ Keep orthogonality: Orthogonal Tucker, HOSVD

 $\mathcal{T} = (\boldsymbol{\mathsf{U}},\boldsymbol{\mathsf{V}},\boldsymbol{\mathsf{W}})\cdot\mathcal{C}$

C is $R_1 \times R_2 \times R_3$: multilinear rank = (R_1, R_2, R_3)

• Keep diagonality: Canonical Polyadic decomposition (CP)

$$\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$$

 \mathcal{L} is $R \times R \times R$ diagonal, $\lambda_i \neq 0$: rank = R.

Pierre Comon

LVA/ICA - Sept. 2010

イロト 不得下 イヨト イヨト 二日

Intro 00 Exact CP

Approx CP

NonNeg 000 Coherence

AAP ENE

Exact Canonical Polyadic (CP) decomposition

Pierre Comon

LVA/ICA - Sept. 2010

イロト イヨト イヨト イヨト

10 / 43

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	0000000	00000	000	0000	000000	000
CP definition							

Canonical Polyadic (CP) decomposition

• Any $I \times J \times \cdots \times K$ tensor \mathcal{T} can be decomposed as

$$\mathcal{T} = \sum_{q} \lambda_q \, \mathbf{u}^{(q)} \otimes \mathbf{v}^{(q)} \otimes \ldots \otimes \mathbf{w}^{(q)}$$

- ➡ "Polyadic form" [Hitchcock'27]
- The tensor rank of T is the minimal number R(T) of "decomposable" terms such that equality holds.
- May impose unit norm vectors $\mathbf{u}^{(q)}, \mathbf{v}^{(q)}, \dots \mathbf{w}^{(q)}$

LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト 二日

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	0000000	00000	000	0000	000000	000
CP definition							

Canonical Polyadic (CP) decomposition

• Any $I \times J \times \cdots \times K$ tensor \mathcal{T} can be decomposed as

$$\mathcal{T} = \sum_{q}^{R(\mathcal{T})} \lambda_q \, \mathbf{u}^{(q)} \otimes \mathbf{v}^{(q)} \otimes \ldots \otimes \mathbf{w}^{(q)}$$

"Polyadic form" [Hitchcock'27]

- The tensor rank of T is the minimal number R(T) of "decomposable" terms such that equality holds.
- May impose unit norm vectors **u**^(q), **v**^(q),... **w**^(q)

LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	0000000	00000	000	0000	000000	000
CP definition							

Canonical Polyadic (CP) decomposition

• Any $I \times J \times \cdots \times K$ tensor \mathcal{T} can be decomposed as

$$\mathcal{T} = \sum_{q}^{R(\mathcal{T})} \lambda_q \, \mathbf{u}^{(q)} \otimes \mathbf{v}^{(q)} \otimes \ldots \otimes \mathbf{w}^{(q)}$$

➡ "Polyadic form" [Hitchcock'27]

- The *tensor rank* of *T* is the minimal number *R*(*T*) of "decomposable" terms such that equality holds.
- May impose unit norm vectors $\mathbf{u}^{(q)}, \mathbf{v}^{(q)}, \dots \mathbf{w}^{(q)}$

LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00 CD 1 C 11	00000	0000000	00000	000	0000	000000	000
CP definiti	on						
			Hitch	rock			



Frank Lauren Hitchcock (1875-1957) [Courtesy of L-H.Lim]

LVA/ICA - Sept. 2010

・ロト ・ 聞 ト ・ ヨト ・ ヨト … ヨ

Intro 00 CP definiti	General 00000 on	Exact CP 0000000	Approx CP 00000	NonNeg 000	Coherence 0000	AAP 000000	END 000		
Hitchcock									



Frank Lauren Hitchcock (1875-1957) [Courtesy of L-H.Lim]



(1916 - 2001)

< □ > < □ > < ⊇ > < ⊇ >
 LVA/ICA - Sept. 2010

Pierre Comon

≣ ≪ 12

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END	
00	00000	00000000	00000	000	0000	000000	000	
CP definition								

Towards a unique terminology

- Minimal Polyadic Form [Hitchcock'27]
- Canonical decomposition [Weinstein'84, Carroll'70, Chiantini-Ciliberto'06, Comon'00, Khoromskij, Tyrtyshnikov]
- Parafac [Harshman'70, Sidiropoulos'00]
- Optimal computation [Strassen'83]
- Minimum-length additive decomposition (AD) [larrobino'96]
 Suggestion:
- Canonical Polyadic decomposition (CP) [Comon'08, Grasedyk, Espig...]
- CP does also already stand for Candecomp/Parafac [Bro'97, Kiers'98, tenBerge'04...]

イロト 不得下 イヨト イヨト 二日

Intro 00 CP definitio	General 00000 n	Exact CP 00000000	Approx CP 00000	NonNeg 000	Coherence 0000	AAP 000000	END 000			
Psychomotrics										





Richard A. Harshman (1970)



J. Douglas Carroll (1970)

LVA/ICA - Sept. 2010

◆□ ▶ ◆□ ▶ ◆臣 ▶ ◆臣 ▶ ─ 臣 ─ のへで

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
CP definit	ion						
					,		

Uniqueness: Kruskal 1/2



The Kruskal rank of a matrix **A** is the maximum number k_A , such that any subset of k_A columns are linearly independent.

LVA/ICA - Sept. 2010

・ロト ・四ト ・ヨト ・ヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
CP definit	ion						

Uniqueness: Kruskal 2/2

Sufficient condition for uniqueness of CP [*Kruskal'77, Sidiropoulos-Bro'00, Landsberg'09*]:

Essential uniqueness is ensured if tensor rank R is below the so-called *Kruskal's bound*:

$$2R + 2 \le k_A + k_B + k_C \tag{3}$$

or *generically*, for a tensor of order d and dimensions N_ℓ :

$$2R \leq \sum_{\ell=1}^d \min(N_\ell, R) - d + 1$$

Pierre Comon
Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	000000000	00000	000	0000	000000	000
CP definit	tion						

Uniqueness: Kruskal 2/2

Sufficient condition for uniqueness of CP [*Kruskal'77, Sidiropoulos-Bro'00, Landsberg'09*]:

Essential uniqueness is ensured if tensor rank R is below the so-called *Kruskal's bound*:

$$2R + 2 \le k_A + k_B + k_C \tag{3}$$

or generically, for a tensor of order d and dimensions N_{ℓ} :

$$2R \leq \sum_{\ell=1}^d \min(N_\ell, R) - d + 1$$

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
CP definition	on						

Rank-3 example 1/2

Example



LVA/ICA - Sept. 2010

・ロト ・四ト ・ヨト ・ヨト

э

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
CP definitio	n						

Rank-3 example 1/2

Example



Pierre Comon

LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	0000000	00000	000	0000	000000	000
CP definit	ion						

Rank-3 example 2/2

Conclusion: the $2 \times 2 \times 2$ tensor

$$\mathcal{T} = 2 \left[\begin{array}{cc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right]$$

admits the CP

$$\mathcal{T} = \left(\begin{array}{c} 1\\1 \end{array}\right)^{\otimes 3} + \left(\begin{array}{c} -1\\1 \end{array}\right)^{\otimes 3} + 2 \left(\begin{array}{c} 0\\-1 \end{array}\right)^{\otimes 3}$$

and has rank 3, hence larger than dimension

LVA/ICA - Sept. 2010

Approximate Canonical Polyadic (CP) decomposition

LVA/ICA - Sept. 2010

イロト イヨト イヨト イヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Motivation							

Why need for approximation?

- Additive noise in measurements
- Noise has a continuous probability distribution
- Then tensor rank is *generic*
- Hence there are often *infinitely many* CP decompositions

Approximations aim at getting rid of noise, and at restoring uniqueness:

$$\operatorname{Arg\,}_{\mathbf{a}(p),\mathbf{b}(p),\mathbf{c}(p)} ||\mathcal{T} - \sum_{p=1}^{r} \mathbf{a}(p) \otimes \mathbf{b}(p) \dots \otimes \mathbf{c}(p)||^{2}$$
(4)

But infimum may be reached for tensors of rank > r !

Pierre Comon

LVA/ICA - Sept. 2010

イロト 不得 トイヨト イヨト

20 / 43

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Motivation							

Why need for approximation?

- Additive noise in measurements
- Noise has a continuous probability distribution
- Then tensor rank is *generic*
- Hence there are often *infinitely many* CP decompositions

➡ Approximations aim at getting rid of noise, and at restoring uniqueness:

$$\operatorname{Arg\,inf}_{\mathbf{a}(p),\mathbf{b}(p),\mathbf{c}(p)} ||\mathcal{T} - \sum_{p=1}^{r} \mathbf{a}(p) \otimes \mathbf{b}(p) \dots \otimes \mathbf{c}(p)||^{2}$$
(4)

But infimum may be reached for tensors of rank > r !

Pierre Comon

LVA/ICA - Sept. 2010

・ロット (雪) (日) (日)

20 / 43

Intro 00 Border rank	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence 0000	AAP 000000	END 000
			Border	rank			

 \mathcal{T} has *border rank* R iff it is the limit of tensors of rank R, and not the limit of tensors of lower rank.

[Bini'79, Schönhage'81, Strassen'83, Likteig'85]



Pierre Comon

Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence 0000	AAP 000000	END 000				
Border ran	¢										
			Exam	ple							

Let **u** and **v** be non collinear vectors. Define \mathcal{T}_0 [Comon et al. '08]:

 $\mathcal{T}_0 = \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$

And define sequence $\mathcal{T}_{\varepsilon} = \frac{1}{\varepsilon} \left[(\mathbf{u} + \varepsilon \mathbf{v})^{\otimes 4} - \mathbf{u}^{\otimes 4} \right]$ Then $\mathcal{T}_{\varepsilon} \to \mathcal{T}_{0}$ as $\varepsilon \to 0$ \blacktriangleright Hence rank $\{\mathcal{T}_{0}\} = 4$, but rank $\{\mathcal{T}_{0}\} = 2$

LVA/ICA - Sept. 2010

Let **u** and **v** be non collinear vectors. Define \mathcal{T}_0 [Comon et al. '08]:

 $\mathcal{T}_0 = \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$

And define sequence $T_{\varepsilon} = \frac{1}{\varepsilon} \left[(\mathbf{u} + \varepsilon \mathbf{v})^{\otimes 4} - \mathbf{u}^{\otimes 4} \right].$ Then $T_{\varepsilon} \to T_0$ as $\varepsilon \to 0$ Hence $\operatorname{rank}\{T_{\varepsilon}\} = 4$, but $\operatorname{rank}\{T_{\varepsilon}\} = 2$

LVA/ICA - Sept. 2010

Let **u** and **v** be non collinear vectors. Define \mathcal{T}_0 [Comon et al.'08]:

 $\mathcal{T}_0 = \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$

And define sequence $\mathcal{T}_{\varepsilon} = \frac{1}{\varepsilon} \left[(\mathbf{u} + \varepsilon \mathbf{v})^{\otimes 4} - \mathbf{u}^{\otimes 4} \right].$ Then $\mathcal{T}_{\varepsilon} \to \mathcal{T}_{0}$ as $\varepsilon \to 0$

▶ Hence rank $\{T_0\} = 4$, but $\underline{rank}\{T_0\} = 2$

LVA/ICA - Sept. 2010

Let **u** and **v** be non collinear vectors. Define \mathcal{T}_0 [Comon et al.'08]:

$$\mathcal{T}_0 = \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$$

And define sequence $\mathcal{T}_{\varepsilon} = \frac{1}{\varepsilon} \left[(\mathbf{u} + \varepsilon \mathbf{v})^{\otimes 4} - \mathbf{u}^{\otimes 4} \right].$ Then $\mathcal{T}_{\varepsilon} \to \mathcal{T}_{0}$ as $\varepsilon \to 0$

▶ Hence $rank{T_0} = 4$, but $rank{T_0} = 2$

Pierre Comon

LVA/ICA - Sept. 2010

22 / 43

Intro 00	General 00000	Exact CP 00000000	Approx CP ○○○●○	NonNeg 000	Coherence 0000	AAP 000000	END 000
Border ran	ik						
			III-nose	dness			
			m-pose	uncss			

Tensors for which $\underline{rank}\{\mathcal{T}\}< rank\{\mathcal{T}\}$ are such that the approximating sequence contains several decomposable tensors which

- tend to infinity and
- cancel each other, viz, some columns become close to collinear

Ideas towards a well-posed problem:

Prevent collinearity or bound columns.

LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト

Intro 00	General 00000	Exact CP 00000000	Approx CP ○○○●○	NonNeg 000	Coherence 0000	AAP 000000	END 000
Border ran	k						
			III noco	dnocc			
			m-pose	uness			

Tensors for which $\underline{rank}\{\mathcal{T}\} < rank\{\mathcal{T}\}$ are such that the approximating sequence contains several decomposable tensors which

- tend to infinity and
- cancel each other, viz, some columns become close to collinear

Ideas towards a well-posed problem:

Prevent collinearity or bound columns.

LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト

Intro 00	General 00000	Exact CP 00000000	Approx CP ○○○●○	NonNeg 000	Coherence 0000	AAP 000000	END 000
Border ran	k						
			III noco	dnocc			
			m-pose	uness			

Tensors for which $\underline{rank}\{\mathcal{T}\}< rank\{\mathcal{T}\}$ are such that the approximating sequence contains several decomposable tensors which

- tend to infinity and
- cancel each other, viz, some columns become close to collinear

Ideas towards a well-posed problem:

Prevent collinearity *or* bound columns.

LVA/ICA - Sept. 2010

ヘロト 人間ト 人目下 人口ト

Intro 00	General 00000	Exact CP 00000000	Approx CP ○○○●○	NonNeg 000	Coherence 0000	AAP 000000	END 000
Border ran	k						
			III noco	dnocc			
			m-pose	uness			

Tensors for which $\underline{rank}\{\mathcal{T}\}< rank\{\mathcal{T}\}$ are such that the approximating sequence contains several decomposable tensors which

- tend to infinity and
- cancel each other, viz, some columns become close to collinear

Ideas towards a well-posed problem:

Prevent collinearity *or* bound columns.

LVA/ICA - Sept. 2010

ヘロト 人間ト 人目下 人口ト

Intro 00 Existence	General 00000	Exact CP 00000000	Approx CP ○○○○●	NonNeg 000	Coherence 0000	AAP 000000	END 000		
Remedies									

I CHICUIC3

- Impose *orthogonality* of columns within factor matrices 1 [Comon'92]
- 2 Impose orthogonality between decomposable tensors [Kolda'01]
- 3 Prevent tendency to infinity by *norm constraint* on factor matrices [Paatero'00]

イロト 不得下 イヨト イヨト 二日

Intro 00 Existence	General 00000	Exact CP 00000000	Approx CP ○○○○●	NonNeg 000	Coherence 0000	AAP 000000	END 000		
Remedies									

ricificules

- Impose *orthogonality* of columns within factor matrices 1 [Comon'92]
- 2 Impose orthogonality between decomposable tensors [Kolda'01]
- Prevent tendency to infinity by norm constraint on factor matrices [Paatero'00]
- 4 Nonnegative tensors: impose decomposable tensors to be nonnegative [Lim-Comon'09] \rightarrow "nonnegative rank"
- 5 Impose minimal angle between columns of factor matrices [Lim-Comon'10]

イロト 不得下 イヨト イヨト 二日

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000

Nonnegativity constraint: example

Pierre Comon

LVA/ICA - Sept. 2010

25 / 43

◆□▶ ◆□▶ ◆ □▶ ◆ □▶ ─ □ ─ つへで



Fluorescence Spectroscopy 1/3

An optical excitation produces several effects

- Rayleigh diffusion
- Raman diffusion
- Fluorescence

At low concentrations, Beer-Lambert law can be linearized [Luciani'09]

$$I(\lambda_{\ell},\lambda_{e},k) = I_{o} \sum_{\ell} \gamma_{\ell}(\lambda_{\ell}) \epsilon_{\ell}(\lambda_{e}) c_{k,\ell}$$

Hence 3rd array decomposition with real nonnegative factors [Bro'97].

LVA/ICA - Sept. 2010

・ロト ・ 四ト ・ ヨト ・ ヨト ・



Fluorescence Spectroscopy 1/3

An optical excitation produces several effects

- Rayleigh diffusion
- Raman diffusion
- Fluorescence

At low concentrations, Beer-Lambert law can be linearized [Luciani'09]

$$I(\lambda_f, \lambda_e, k) = I_o \sum_{\ell} \gamma_{\ell}(\lambda_f) \epsilon_{\ell}(\lambda_e) c_{k,\ell}$$

Hence 3rd array decomposition with real *nonnegative* factors [Bro'97].

Pierre Comon

LVA/ICA - Sept. 2010

< ロ > < 同 > < 回 > < 回 > < □ > <

26 / 43



Fluorescence Spectroscopy 1/3

An optical excitation produces several effects

- Rayleigh diffusion
- Raman diffusion
- Fluorescence

At low concentrations, Beer-Lambert law can be linearized [Luciani'09]

$$I(\lambda_f, \lambda_e, k) = I_o \sum_{\ell} \gamma_{\ell}(\lambda_f) \epsilon_{\ell}(\lambda_e) c_{k,\ell}$$

➡ Hence 3rd array decomposition with real *nonnegative* factors [Bro'97].

LVA/ICA - Sept. 2010

イロト 不得下 イヨト イヨト



Fluorescence Spectroscopy 2/3

Mixture of 4 solutes (one concentration shown)



Pierre Comon

LVA/ICA - Sept. 2010

・ロト ・ 四ト ・ ヨト ・ ヨト ・

27 / 43



Fluorescence Spectroscopy 3/3



Obtained results with R = 4

Pierre Comon

LVA/ICA - Sept. 2010

itro Gen 0 000 Exact CP 00000000 Approx CP

NonNeg 000 Coherence

00 000

AAP

Approximate CP decomposition: Coherence constraint

Pierre Comon

LVA/ICA - Sept. 2010

・ロト ・ 四ト ・ ヨト ・ ヨト ・

29 / 43

Intro 00	General	Exact CP	Approx CP	NonNeg	Coherence	AAP 000000	END 000
Coherence							
			C 1				
			Cohere	ence			

 Definition: [Donoho'03, Gribonval'03, Candès'07] let A a matrix with unit-norm columns, a_p.

$$\mu_{A} = \max_{p \neq q} |\langle \mathbf{a}_{p}, \mathbf{a}_{q} \rangle| \tag{5}$$

■ this "coherence of A" is used in Sparse Representation theory

LVA/ICA - Sept. 2010

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence ○●○○	AAP 000000	END 000		
Existence:	coercivity								
Existence									

Best rank-R approximate under angular constraint

■ **Proposition:** [Lim-Comon'2010]

Let \mathcal{L} diagonal, and **A**, **B** and **C** have R unit norm columns. If $R < [\mu_A \mu_B \mu_C]^{-1}$, then:

$$\inf_{\mathbf{A},\mathbf{B},\mathbf{C}} ||\mathcal{T} - (\mathbf{A},\mathbf{B},\mathbf{C}) \cdot \mathcal{L}||$$

is attained.

LVA/ICA - Sept. 2010

Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence	AAP 000000	END 000
Uniqueness							

Uniqueness: Back to Kruskal

Lemma: (spark) [Gribonval'03]

$$k_A \ge \frac{1}{\mu_A}$$

Pierre Comon

LVA/ICA - Sept. 2010

32 / 43

(ロ)、(型)、(E)、(E)、(E)、(O)への



Uniqueness

■ Proposition: [Lim-Comon'10] If $\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$, with $\lambda_p \neq 0$ for $1 \leq p \leq R$, **A**, **B**, **C** matrices with unit norm columns, and:

$$2R + 2 \le \frac{1}{\mu_A} + \frac{1}{\mu_B} + \frac{1}{\mu_C}$$
(6)

then T has a unique CP decomposition of rank R, up to unit modulus scalar factors (ρ_A, ρ_B, ρ_C), $\rho_A \rho_B \rho_C = 1$.

Hence it suffices that one µ is small, not every

Pierre Comon

LVA/ICA – Sept. 2010

イロト 不得下 イヨト イヨト



Uniqueness

■ Proposition: [Lim-Comon'10] If $\mathcal{T} = (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathcal{L}$, with $\lambda_p \neq 0$ for $1 \leq p \leq R$, **A**, **B**, **C** matrices with unit norm columns, and:

$$2R + 2 \le \frac{1}{\mu_A} + \frac{1}{\mu_B} + \frac{1}{\mu_C}$$
(6)

then T has a unique CP decomposition of rank R, up to unit modulus scalar factors (ρ_A, ρ_B, ρ_C), $\rho_A \rho_B \rho_C = 1$.

• Hence it suffices that one μ is small, not every

Pierre Comon

LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	0000000	00000	000	0000	000000	000

Angular constraint: example

Pierre Comon

LVA/ICA - Sept. 2010

34 / 43

◆ロト ◆昼 → ◆臣 → ◆臣 → ○ ● ○ ● ● ●





Pierre Comon

LVA/ICA - Sept. 2010

35 / 43

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	00000	000
MIMO							



LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	00000	000
MIMO							



LVA/ICA - Sept. 2010

・ロト ・四ト ・ヨト ・ヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	00000	000
MIMO							



LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	00000	000
MIMO							



イロト イポト イヨト イヨト
Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	00000	000
MIMO							

Antenna Array Processing



LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト



Narrow band model in the far field

Modeling the signals received on an array of antennas generally leads to a *matrix decomposition*:

$$T_{ij} = \sum_{q} a_{iq} s_{jq}$$

i: spaceq: path, sourceA: antenna gainsj: timeS: transmitted signals

But in the presence of additional *diversity*, a tensor can be constructed, thanks to new index *p*

LVA/ICA - Sept. 2010

イロト イポト イヨト イヨト



Narrow band model in the far field

Modeling the signals received on an array of antennas generally leads to a *matrix decomposition*:

$$T_{ijp} = \sum_{q} a_{iq} \, s_{jq} \, h_{pq}$$

i: space *q*: path, source **A**: antenna gains *j*: time **S**: transmitted signals But in the presence of additional *diversity* a tensor can be

But in the presence of additional *diversity*, a tensor can be constructed, thanks to new index p

LVA/ICA - Sept. 2010

< ロ > < 同 > < 回 > < 回 > < □ > <

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	0000000	00000	000	0000	000000	000
MIMO							

Possible diversities in Signal Processing

space

time

- space translation (array geometry)
- time translation (chip)
- frequency/wavenumber (nonstationarity)
- excess bandwidth (oversampling)
- cyclostationarity
- polarization
- finite alphabet

...

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Geometry							
		Cnace +r	andation	diversity	(1/2)		
		Space tr	ansiation	aiversity	Y (1/2)		



・ロト・(型ト・(ヨト・(型ト・(ロト

Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg	Coherence	AAP 000000	END 000
Geometry							
		-					
		Space tr	anslation	diversity	y (1/2)		



Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence 0000	AAP 000000	END 000
Geometry							
		Space tr	andation	divorcity	(1/2)		
		Space tr	ansiation	uiversity	y (1/2)		



Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Geometry							
		Space tr	andation	divorcity	(1/2)		
		Space th	ansiation	uiversity	(1/2)		



▲□▶▲圖▶▲≣▶▲≣▶ ≣ のへの





イロト イヨト イヨト イヨト

Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence 0000	AAP 000000	END 000
Geometry							
		Space tr	andation	divorcity	(1/2)		
		Space tr	ansiation	uiversity	y (1/2)		



Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence	AAP ○○○●○○	END 000
Geometry							
		-			(
		Space tr	anslation	diversity	y (1/2)		



・ロト・(型ト・(ヨト・(型ト・(ロト

Intro 00	General 00000	Exact CP 00000000	Approx CP	NonNeg 000	Coherence 0000	AAP ○○○●○○	END 000		
Geometry									
		-			(
Space translation diversity $(1/2)$									



・ロト・日本・日本・日本・日本・

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Geometry							

Space translation diversity (2/2)

 A_{iq} : gain between sensor *i* and source *q* H_{pq} : transfer between reference and subarray *p* S_{jq} : sample *j* of source *q* β_q : path loss, **d**_q: DOA, **b**_i: sensor location

Tensor model (NB far field) [Sidiropoulos'00]

- Reference subarray: $A_{iq} = \beta_q \exp(j\frac{\omega}{C} \mathbf{b}_i^{\mathsf{T}} \mathbf{d}_q)$
- Space translation (from reference subarray) $\beta_q \exp(\frac{\omega}{C} [\mathbf{b}_i + \mathbf{\Delta}_p]^T \mathbf{d}_q) \stackrel{\text{def}}{=} A_{iq} H_{pq}$ = Trilineer model:

Irilinear model:

$$T_{ijp} = \sum_{q} A_{iq} S_{jq} H_{pq}$$

p: index of subarray

Pierre Comon

LVA/ICA - Sept. 2010

◆□▶ ◆圖▶ ◆臣▶ ◆臣▶ ○臣

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Geometry							

Space translation diversity (2/2)

 A_{iq} : gain between sensor *i* and source *q* H_{pq} : transfer between reference and subarray *p* S_{jq} : sample *j* of source *q* β_{q} : path loss, **d**_q: DOA, **b**_i: sensor location

Tensor model (NB far field) [Sidiropoulos'00]

- Reference subarray: $A_{iq} = \beta_q \exp(j\frac{\omega}{C} \mathbf{b}_i^{\mathsf{T}} \mathbf{d}_q)$
- Space translation (from reference subarray):

$$\beta_q \exp\left(\frac{\omega}{C} \left[\mathbf{b}_i + \mathbf{\Delta}_p\right]^{\mathsf{T}} \mathbf{d}_q\right) \stackrel{\text{def}}{=} A_{iq} H_{pq}$$

Trilinear model:

$$T_{ijp} = \sum_{q} A_{iq} \, S_{jq} \, H_{pq}$$

p: index of subarray

Pierre Comon

LVA/ICA - Sept. 2010

・ロト ・聞ト ・ヨト ・ヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Geometry							

Space translation diversity (2/2)

 A_{iq} : gain between sensor *i* and source *q* H_{pq} : transfer between reference and subarray *p* S_{jq} : sample *j* of source *q* β_{q} : path loss, **d**_q: DOA, **b**_i: sensor location

Tensor model (NB far field) [Sidiropoulos'00]

- Reference subarray: $A_{iq} = \beta_q \exp(j\frac{\omega}{C} \mathbf{b}_i^{\mathsf{T}} \mathbf{d}_q)$
- Space translation (from reference subarray):

$$\beta_q \exp(\frac{\omega}{C} [\mathbf{b}_i + \mathbf{\Delta}_p]^{\mathsf{T}} \mathbf{d}_q) \stackrel{\text{def}}{=} A_{iq} H_{pq}$$

Trilinear model:

$$T_{ijp} = \sum_{q} A_{iq} \, S_{jq} \, H_{pq}$$

p: index of subarray

Pierre Comon

LVA/ICA - Sept. 2010

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Geometry							

Meaning of coherence

- Unconstrained joint source estimation & localization
 [Sidiropoulos'2000] (ill-posed if approximate)
- Coherence-constrained joint source estimation & localization [Lim-Comon'2010]
 - time diversity: $\mu_A \rightarrow cross \ correlation$
 - space diversity: μ_{B} , $\mu_{C} \rightarrow$ angular separation



Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END		
00	00000	0000000	00000	000	0000	000000	000		
Yet other unaddressed topics									

Unaddressed topics

- spread spectrum communications
- brain inverse problems
- medical imaging (MRI)
- NL filtering
- noise reduction
- compression (Tensor trains...)
- probability
- hyperspectral imaging
- structured tensors
- convolutive mixtures
- nonnegative factors
- • •
- Algorithms

د ت د د د LVA/ICA - Sept. 2010

2008

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END		
00	00000	00000000	00000	000	0000	000000	000		
Why use tensors									

Main reason: essential uniquenessIdentifiability recovery, up to scale-permutation

Sometimes: powerful deterministic approaches

Secondary reason: more sources with fewer sensors
 Matrices A, B, C may have more columns than rows



・ロト ・四ト ・ヨト ・ヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END		
00	00000	00000000	00000	000	0000	000000	000		
Why use tensors									

Main reason: essential uniqueness ➡ Identifiability recovery, up to scale-permutation

Sometimes: powerful deterministic approaches

Secondary reason: more sources with fewer sensors
 Matrices A, B, C may have more columns than rows



イロト イポト イヨト イヨト

00 00000 000000 0000 000 0000 0000 00000	Intro	A	Exact CP	Approx CP	NonNeg	Coherence	AAP	E	ND
Why use tensors	00	С	0000000	00000	000	0000	000000	0	•0

Main reason: essential uniqueness

➤ Identifiability recovery, up to scale-permutation

Sometimes: powerful deterministic approaches

Secondary reason: more sources with fewer sensors ➡ Matrices A, B, C may have more columns than rows



イロト イポト イヨト イヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END		
00	00000	00000000	00000	000	0000	000000	000		
Why use tensors									

Main reason: essential uniqueness

Identifiability recovery, up to scale-permutation

Sometimes: powerful deterministic approaches

Secondary reason: more sources with fewer sensors
 Matrices A, B, C may have more columns than rows



イロト イポト イヨト イヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END
00	00000	00000000	00000	000	0000	000000	000
Why use t	tensors						

Other perspectives

Ignorance is the necessary condition for human being happiness. Anatole France (1844-1924)



イロト イポト イヨト イヨト

Intro	General	Exact CP	Approx CP	NonNeg	Coherence	AAP	END		
00	00000	00000000	00000	000	0000	000000	000		
Why use tensors									

Other perspectives

Ignorance is the necessary condition for human being happiness. Anatole France (1844-1924)



イロト イポト イヨト イヨト



Only when the last tree has died, the last river has been poisoned and the last fish has been caught will we realize that we cannot eat money. Cree proverb